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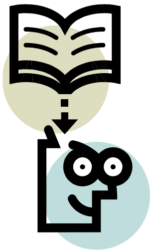
StatEase®
statistics made easy®

Handbook for Experimenters



A concise collection of handy tips
to help you set up and analyze
your designed experiments.

Version 23.00

Acknowledgements

The statistical staff at Stat-Ease thank the countless colleagues and clients who helped us improve our software, training and consulting for design of experiments (DOE). Keep the ideas coming!

NEED HELP WITH A DOE?

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Introduction to the Stat-Ease *Handbook for Experimenters*

Design of experiments (DOE) provides an incredibly powerful research tool by which you purposefully change input factors to observe their impact on your process or formulation. DOE has been embraced by every high-tech industry on the planet—chemical, pharmaceutical, electronics, automotive, etc.

Traditional scientific methods require that experimenters change only one-factor-at-a-time (OFAT). However, OFAT falls far short of multifactor DOE for efficiency in measuring main effects and effectiveness for detecting breakthrough interactions.

We designed this handbook for our software—Design-Expert® (DX) and Stat-Ease® 360 (SE360) users. Refer to Section 1 before designing your experiment. Section 2 helps you analyze your resulting experimental data. Section 3 provides statistical tables for your reference (do not worry—as a Stat-Ease software user you will never need them other than to check its calculations).

-The Stat-Ease Consulting Team

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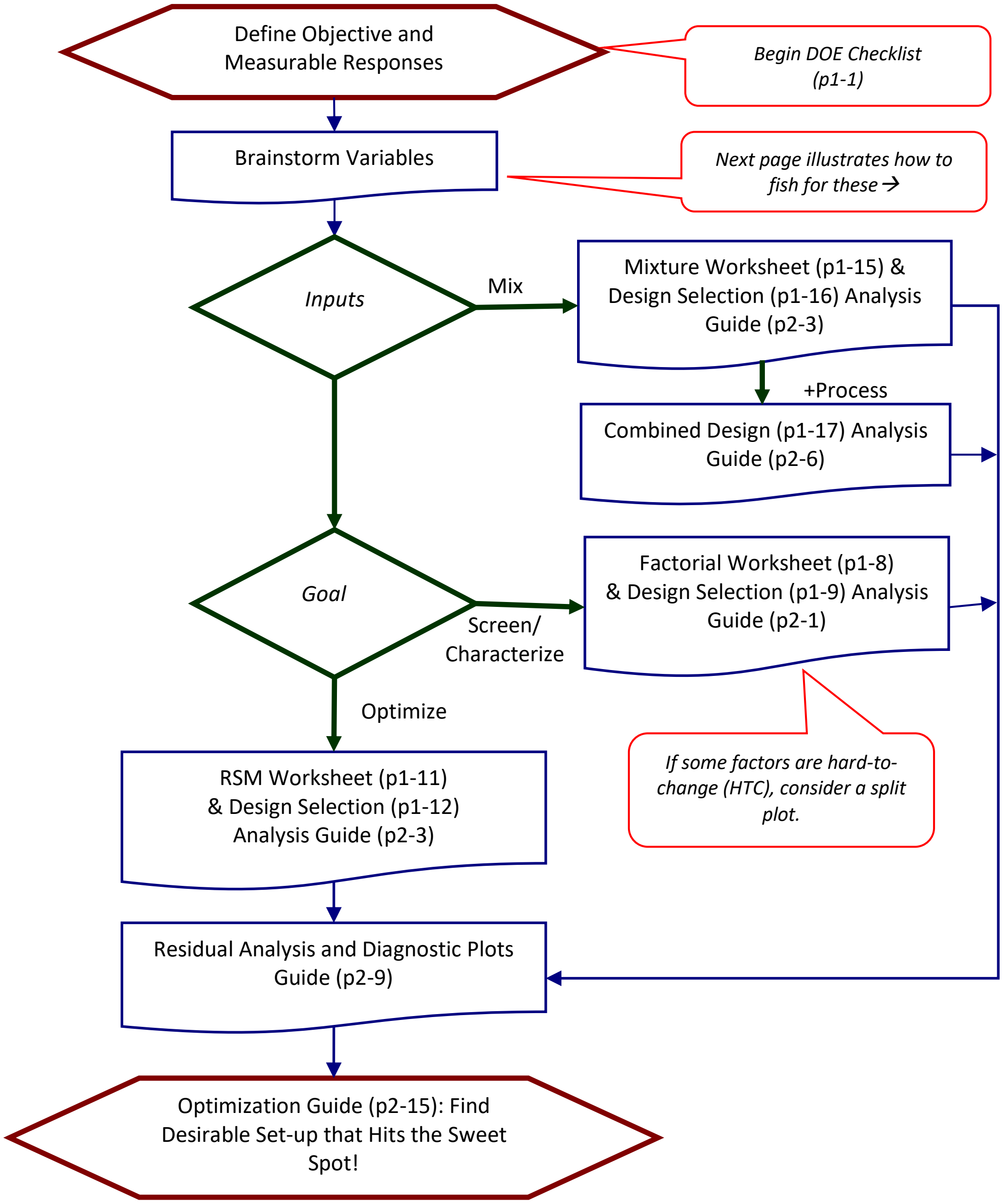
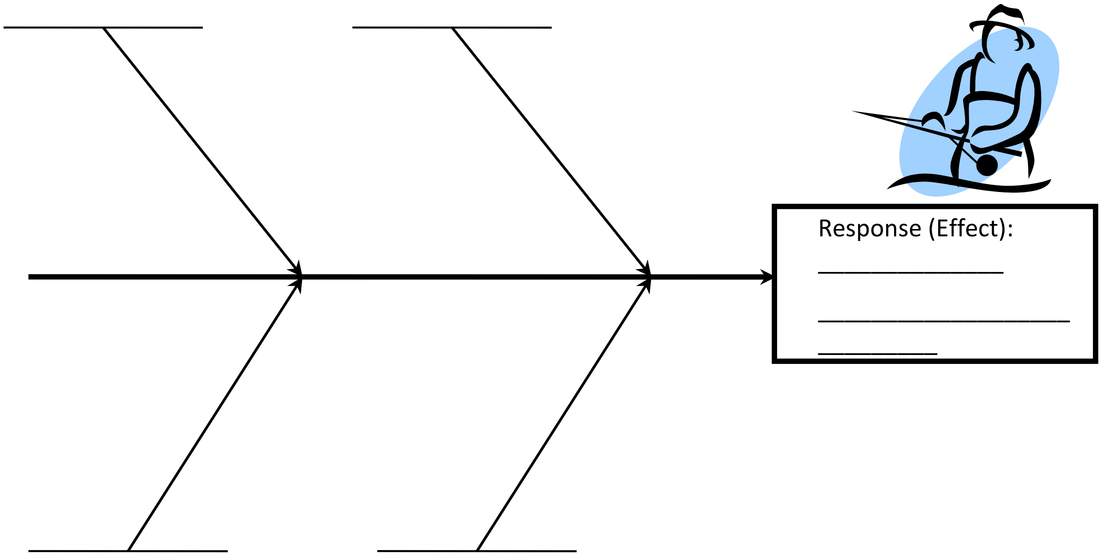
DOE Process Flowchart



Cause-and-effect diagram (to fish for factors)



<u>Suggestions for being creative on identifying potential variables</u>:

- Label the five big fish bones by major causes, for example, *Material*, *Method*, *Machinery*, *People* and *Environment* (spine).
- Gather a group of subject matter experts, as many as a dozen, and
  - Assign one to be leader, who will be responsible for maintaining a rapid flow of ideas.
  - Another individual should record all the ideas as they are presented.
    - Alternative: To be more participative, start by asking everyone to note variables on sticky notes that can then be posted on the fishbone diagram.

<u>Choosing variables to experiment on and what to do with the others</u>:

- For the sake of efficiency, pare the group down to three or so key people who can then critically evaluate the collection of variables and chose ones that would be most fruitful to experiment on.
  - Idea for prioritizing variables: Give each evaluator 100 units of imaginary currency to ‘invest’ in their favorites. Tally up the totals from top to bottom.
- Note factors that are hard to change. Consider either blocking them out or including them for effects assessment via a split plot design.
- Variables not chosen should be held fixed if possible.
- Keep a log of other variables that cannot be fixed but can be monitored.

“It is easier to tone down a wild idea than to think up a new one.”

- Alex Osborne

Section 1:

Designing Your Experiment

DOE Checklist

- ☐ Define objective of the experiment.
- ☐ Identify response variables and how they will be measured.
- ☐ Decide which variables to investigate (brainstorm—see fishbone in Preface).
- ☐ Choose low and high level of each factor (or component if a mixture).
  - Be bold but avoid regions that might be bad or unsafe.
- ☐ Choose a model based on subject matter knowledge.
- ☐ Select design. Specify:
  - Replicates.
  - Blocks (to filter out known source of variation, e.g., day-to-day differences).
  - Center points for characterization factorial or RSM (centroid if a mixture).
- ☐ Evaluate design:
  - Factorials:
    - If fractional, check aliasing of main effect and two-factor interactions.
    - Assess power—aim for 80% at the least.
  - Response surface method (RSM) and/or mixture:
    - Size by fraction of design space (FDS).
- ☐ Go over details of the physical setup and design execution. Be safe!
- ☐ Determine how to hold non-DOE variables constant.
- ☐ Identify uncontrolled variables: Can they be monitored?
- ☐ Negotiate time, material and budgetary constraints.
    - Invest no more than one-quarter of your experimental budget (time and money) in the first design. Take a sequential approach. Be flexible!
- ☐ Discuss any other special considerations for this experiment.
- ☐ Make plans for follow-up studies.
- ☐ Perform confirmation runs.

Factorial DOE Planning Process

This four-step process guides you to an appropriate factorial DOE. Based on a projected signal-to-noise ratio, you will determine how many runs to budget.

1. Identify opportunity and define objective.
2. State objective in terms of measurable responses.
    a. Define the minimum change (Δy) that is important to detect for each response. This is your "signal."
    b. Estimate experimental error (σ) for each response. This is your "noise."
    c. Use the signal to noise ratio ($\Delta y/\sigma$) to estimate power.

    This information is needed for EACH response. See the next page for an example on how to calculate signal to noise and details on power.
3. Select the input factors to study. (Remember that the factor levels chosen determine the size of Δy.)

    The factor ranges must be large enough to (at a minimum) generate the hoped-for change(s) in the response(s).
4. Select a factorial design (see program Help for details).
    - Are any factors hard-to-change (HTC)? If so, consider a split-plot design.
    - If fractionated and/or blocked, evaluate aliases with the order set to a two-factor interaction (2FI) model.
    - Evaluate power (ideally greater than 80%). If the design is a split-plot, consider the trade-off in power versus running a completely randomized experiment.
    - Examine the design to ensure all the factor combinations are reasonable and safe (no disasters!)

Power Requirements for Two-Level Factorials

Purpose:

Determine how many runs you need to achieve at least an 80% chance (power) of revealing an active effect (signal) of size delta (Δ).

General Procedure:

1. Determine the signal delta (Δ). This is the change in the response that you want to detect. Bounce numbers off your management and/or clients, starting with a ridiculously low improvement in the response and working up from there. What's the threshold value that arouses interest? That's the minimum signal you need to detect. Just estimate it the best you can—try something!
2. Estimate the standard deviation sigma (σ)—the noise—from:
   - repeatability studies
   - control charts (R-bar divided by d_2)
   - analysis of variance (ANOVA) from a DOE.
   - historical data or experience (just make a guess!).
3. Set up your design and evaluate its power based on the signal-to-noise ratio (Δ/σ). If it's less than 80%, consider adding more runs or even replicating the entire design.* Continue this process until you achieve the desired power. If the number of runs for a 'right-sized' (adequately powered) design exceed what you can afford, then you must find a way to decrease noise (σ), increase the signal (Δ), or both.

   **(If your initial design is fractional factorial, then do not simply replicate the runs, instead chose a less-fractional design, thus improving power and resolution.)*

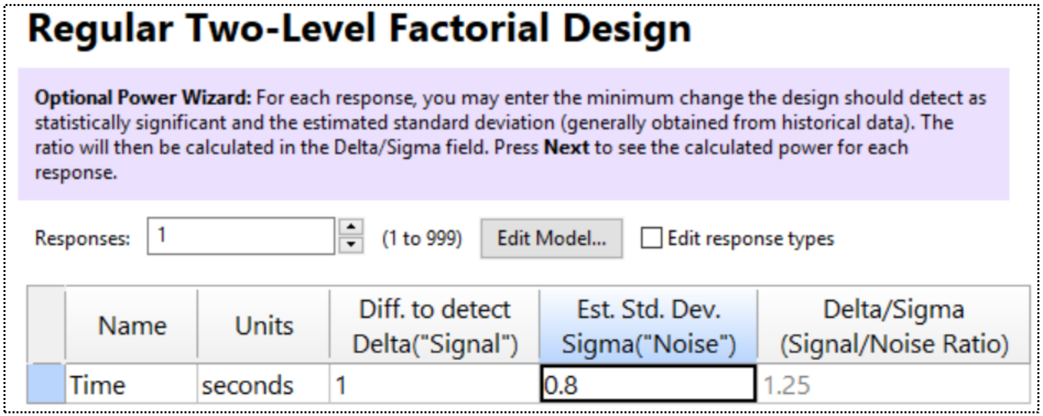
Example—Readability of Avionics Video Display Terminals (VDTs):

What is the ideal color/typeface combination to maximize readability of video display terminals? A team of ergonomic engineers at an avionics manufacturer want to set up an experiment on three VDT factors: A. Foreground (black or yellow), B. Background (white or cyan) and C. typeface (Arial or Times New Roman). A full 2^3 factorial design (8 runs) is set up with the goal to minimize response time needed to read a 30-word paragraph. The signal-to-noise ratio is calculated as follows:

- A 1-second improvement is the smallest value that arouses interest from the client. This is the signal: $\Delta = 1$.
- A prior DOE reveals a standard deviation of 0.8 seconds in readings. This is the noise: $\sigma = 0.8$.
- The signal to noise ratio (Δ/σ) is 1/.8 = 1.25. The power to detect this should be at least 80%.

Using Stat-Ease Software to Size the VDT Experiment:

1. For an 8-run regular 2^3 design, enter the delta and sigma. The program then computes the signal-to-noise (Δ/σ) ratio of 1.25.

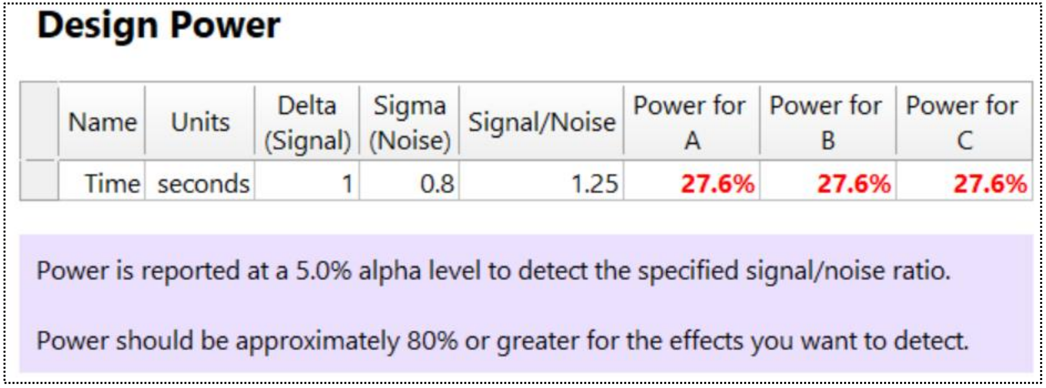


Regular Two-Level Factorial Design

Optional Power Wizard: For each response, you may enter the minimum change the design should detect as statistically significant and the estimated standard deviation (generally obtained from historical data). The ratio will then be calculated in the Delta/Sigma field. Press **Next** to see the calculated power for each response.

Responses: 1 (1 to 999)

Name	Units	Diff. to detect Delta("Signal")	Est. Std. Dev. Sigma("Noise")	Delta/Sigma (Signal/Noise Ratio)
Time	seconds	1	0.8	1.25



Design Power

Name	Units	Delta (Signal)	Sigma (Noise)	Signal/Noise	Power for A	Power for B	Power for C
Time	seconds	1	0.8	1.25	27.6%	27.6%	27.6%

Power is reported at a 5.0% alpha level to detect the specified signal/noise ratio.

Power should be approximately 80% or greater for the effects you want to detect.

The probability of detecting a 1 second difference at the 5% alpha threshold level for significance (95% confidence) is only 27.6%, which falls far short of the desired 80%.

2. Go back and add a 2nd replicate (blocked) to the design (for a total of 16 runs) and re-evaluate the power.

Replicates: 2 Blocks: 2

The power increases to 62.5% for the 1.25 signal/noise ratio – not good enough. ☹

3. Add a 3rd replicate (blocked) to the design (for a total of 24 runs) and evaluate.

Power is now over 80% for the ratio of 1.25:
Mission accomplished! ☺

Impact of **Split-Plot** (vs Randomized) Design on Power:

Illustration:

Engineers need to determine the cause of drive gears becoming ‘dished’ (a geometric distortion). Three of the five suspect factors are hard to change (HTC). To accommodate these HTC factors in a reasonable number of runs, they select a 16-run split-plot regular two-level design and assess the power for a signal of 5 and a noise of 2, with the ratio of whole-plot to split-plot variance at the default of 1.

Stat-Ease software then produces these power calculations:

- The easy-to-change (ETC) factors D and E (capitalized) increase in power (from 88.9% to 98.4%) due to being in the “subplot” of the split-plot design. ☺
- However, the HTC factors a, b and c (lower-case) lose power by being restricted in their randomization to “whole plots”, falling from 88.9% to 58.8%. ☹

Fortunately, subject matter knowledge for this example indicates that the HTC factors vary far less—by a 1-to-4 ratio—than the ETC. The experimenters therefore decrease the variance ratio from 1 to 0.25. This restores adequate power—85.7% (the benchmark being 80%)—to the HTC factors. ☺

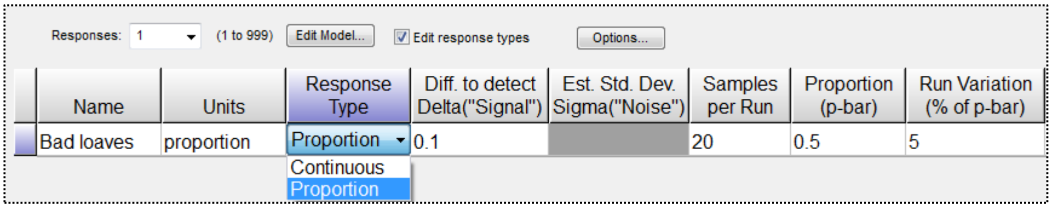
Procedure for Handling Response in **Proportions**:

Illustration:

A small bakery develops a new type of bread that their customers love. Unfortunately, only half of the loaves come out saleable—the remainder falling flat. Perhaps switching to a premium flour (expensive!) and/or making other changes to ingredients, e.g. yeast, might help. The master baker sets up a two-level factorial design for 5 factors in 16 runs, i.e., a high-resolution half-fraction. He figures on baking 20 loaves per run. Here are the steps taken to develop adequate power for this experiment.

1. Convert the measurement to a proportion ("p"), where p = (#of fails or passes) / (#total units).
2. Check (✓) on the **Edit response types**.
3. Determine your current proportion ("p-bar") and the difference ("signal") you want to detect. In this case p-bar is 0.5 (half being failures). The baker decides that it would be good to know if changing the factors can produce a change the proportion of a 10 percent or more. The signal is entered as a fraction of 0.1.
4. Decide a starting point for the "samples per run"—20 being the number for this case.
5. Estimate the run-to-run variation as a percent of the current proportion, assuming a very large number of parts were to be produced at each setup. In this case, 5% of p-bar is the estimate.

Here is Stat-Ease software's power-wizard entry screen for the bread-baking experiment:



Responses: 1 (1 to 999) ☑ Edit response types

Name	Units	Response Type	Diff. to detect Delta("Signal")	Est. Std. Dev. Sigma("Noise")	Samples per Run	Proportion (p-bar)	Run Variation (% of p-bar)
Bad loaves	proportion	Proportion	0.1		20	0.5	5

The proportion response power comes out to be 35.3%: not enough (80% recommended). (Disregard the warning about not changing the default.) This takes the air out of the baker (pun intended) but his spirits rise (ha ha) when he goes back and chooses the full factorial, i.e., 32 runs—this raising the power to 66.3%. Almost there! The baker comes up with a way to squeeze more loaves into the oven and sees his way clear to increasing the samples per run to 30. That does the trick: power increasing to 82.2%. ☺

Special Procedure for Handling **Standard Deviation**

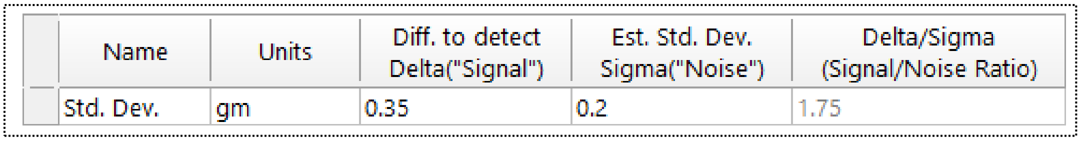
In many situations, you will produce a number (n) of parts or samples per run in your experiment. Then we recommend you compute the standard deviation of each response so you can find robust operating conditions by minimizing variability. If you go this route, we advise an n of 5 to 10 to get a decent estimate of variation. The more parts or samples per run the better, but with diminishing returns—there being little value in going beyond an n of 20.

The standard power calculations for two-level factorials will work in this case, but you must come up with an estimate of the standard deviation of the within-run variability.

Illustration:

When filling packages in the food industry, manufacturers must put in at least the amount listed on label. By minimizing the variability in package weight, specifications can be tightened closer to the stated label amount weight, thus saving money without shorting consumers (and risking costly penalties imposed by regulatory authorities!).

For example, let's say that at current operating conditions for the packager, the fill-to-fill standard deviation is about 1.2 grams (gm). At a minimum, a 0.35 gm change in the standard deviation would be an important difference. The standard deviation from run-to-run varies, of course. Over time the filler is shut down and started up a number for times, from which the food-processing engineer calculates a standard deviation of 0.2 in the fill-to-fill variations. Thus, Stat-Ease software's power wizard entry is:



Name	Units	Diff. to detect Delta("Signal")	Est. Std. Dev. Sigma("Noise")	Delta/Sigma (Signal/Noise Ratio)
Std. Dev.	gm	0.35	0.2	1.75

For a two-level factorial design with 16 runs, this produces a power above 88%—plenty good. ☺ Note that the sigma entered is 0.2—not 1.2. This incorrect level of noise, being many-fold higher, would require hundreds of runs to overpower. Do not make this mistake when calculating power for a response that is the standard deviation of your response.

Factorial Design Worksheet

Identify opportunity and define objective: ____

____

____

State objective in terms of measurable responses:

- *Define the change (Δ y - signal) you want to detect.*
- *Estimate the experimental error (σ - noise)*
- *Use Δ y⁄σ (signal to noise) to check for adequate power.*

Name	Units	Δ y	σ	Δ y/ σ	Power	Goal*
R_1:						
R_2:						
R_3:						
R_4:						

*Goal: minimize, maximize, target=x, etc.

Select the input factors and ranges to vary within the experiment:

Remember that the factor levels chosen determine the size of Δ y.

Name	Units	Type	HTC*?	Low (−1)	High (+1)
A:					
B:					
C:					
D:					
E:					
F:					
G:					
H:					
J:					
K:					

**Hard-to-change (versus easy-to-change—ETC)*

Choose a design: Type:____________________________________

Replicates: ____, Blocks: _____, Center points: ____

Factorial Design Selection

Regular Two-Level: Selection of full and fractional factorial designs where each factor is run at 2 levels. These designs are color-coded in Stat-Ease software to help you identify their nature at-a-glance.

- ☐ White: Full factorials (*no aliases*). All possible combinations of factor levels are run. Provides information on all effects.
- 🟩 Green: Resolution V designs or better (*main effects (ME’s) aliased with four factor interactions (4FI) or higher and two-factor interactions (2FI’s) aliased with three-factor interactions (3FI) or higher.*) Good for estimating ME’s and 2FI’s. Careful: If you block, some 2FI’s may be lost!
- 🟨 Yellow: Resolution IV designs (*ME’s clear of 2FI’s, but these are aliased with each other [2FI – 2FI].)* Useful for screening designs where you want to determine main effects and the existence of interactions.
- 🟥 Red: Resolution III designs (*ME’s aliased with 2FI’s.*) Good for ruggedness testing where you hope your system will not be sensitive to the factors. This boils downs to a go/no-go acceptance test. Caution: Do not use these designs to screen for significant effects.

Min-Run Characterize (Resolution V): Balanced (equireplicated) two-level designs containing the minimum runs to estimate all ME’s and 2FI’s. Check the power of these designs to make sure they can estimate the size effect you need. Caution: If any responses go missing, then the design degrades to Resolution IV.

Irregular Res V: These fractional Resolution V “Legacy” designs, which are not powers of two (e.g.; 4 factors in 12 runs) provide alternatives to the standard full or Res V two-level factorial designs.

Min-Run Screen (Resolution IV): Estimates main effects only *(the 2FI’s remain aliased with each other)*. Check the power. Caution: even one missing run or response degrades the aliasing to Resolution III. To avoid this sensitivity, accept the Stat-Ease software default that adds two extra runs (Min Run +2).

Plackett-Burman: A “Miscellaneous” design suited only for ruggedness testing due to complex Resolution III aliasing. Not good for screening.

Taguchi OA (Orthogonal Array): A “Miscellaneous” Resolution III design typically run saturated - all columns used for ME’s. ‘Linear graphs’ lead to estimating certain interactions. We recommend you not use these designs.

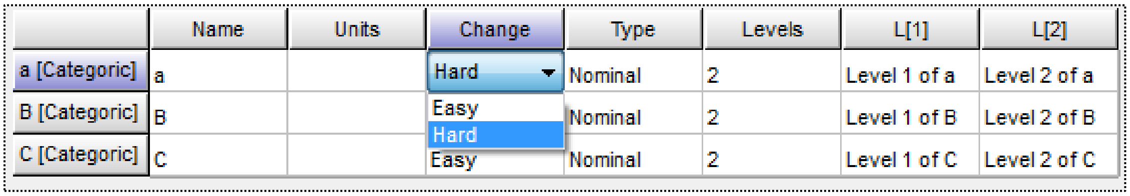
Multilevel Categoric: A general factorial design good for categoric factors with any number of levels: Provides all possible combinations. If too many runs, use Optimal design. *(Design also available in Split-Plot.)*

Optimal (Custom): Choose any number of levels for each categoric factor. The number of runs chosen will depend on the model you specify (2FI by default). D-optimal factorial designs are recommended. *(Optimal designs also available in Split-Plot.)*

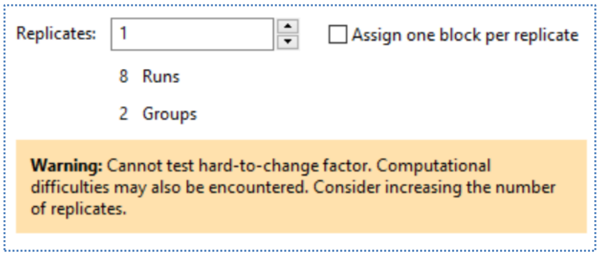
<u>**Split-Plot**</u> <u>Designs:</u>

Regular Two-Level: Select the number of total factors and how many of these will be hard to change (HTC). The program may then change the number of runs to provide power.* The HTCs will be grouped in whole plots, within which the easy-to-change (ETC) factors will be randomized in subplots. From one group to the next, be sure to reset each factor level even if by chance it does not change.
**(<u>Caution</u>: You may be warned on the power screen that "Whole-plot terms cannot be tested…" Proceed then with caution—accepting there being no test on HTC(s)—or go back and increase the runs.)*

Multilevel Categoric: Change factors to Hard or Easy as shown. If you see the "Cannot test…" warning upon Continue, then increase Replicates.



	Name	Units	Change	Type	Levels	L[1]	L[2]
a [Categoric]	a		Hard	Nominal	2	Level 1 of a	Level 2 of a
B [Categoric]	B		Easy Hard	Nominal	2	Level 1 of B	Level 2 of B
C [Categoric]	C		Easy	Nominal	2	Level 1 of C	Level 2 of C



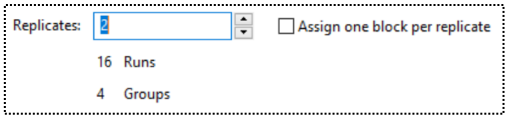
Replicates: 1

☐ Assign one block per replicate

8 Runs

2 Groups

Warning: Cannot test hard-to-change factor. Computational difficulties may also be encountered. Consider increasing the number of replicates.



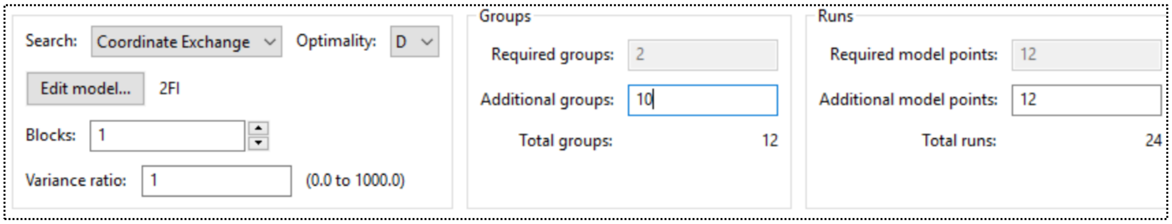
Replicates: 2

☐ Assign one block per replicate

16 Runs

4 Groups

Optimal (Custom): Change factors to Hard or Easy. Watch out for low power on the HTC factor(s). In that case go Back and add more Groups as shown below. Variance ratio (whole plot to subplot) of 1 is a balance that will work for most cases.



Search: Coordinate Exchange

Optimality: D

2FI

Blocks: 1

Variance ratio: 1 (0.0 to 1000.0)

Groups

Required groups: 2

Additional groups: 10

Total groups: 12

Runs

Required model points: 12

Additional model points: 12

Total runs: 24

Response Surface Method (RSM) Design Worksheet

Identify opportunity and define objective: ____

____

____

____

State objective in terms of measurable responses:

- *Define the precision (d - signal) required for each response.*
- *Estimate the experimental error (σ - noise) for each response.*
- *Use d/σ (signal to noise) to check for adequate precision using FDS.*

Name	Units	d	Σ	FDS	Goal
R_1:					
R_2:					
R_3:					
R_4:					

Select the input factors and ranges to vary within the experiment:

Name	Units	Type	Low	High
A:				
B:				
C:				
D:				
E:				
F:				
G:				
H:				

Quantify any MultiLinear Constraints (MLC's):

____

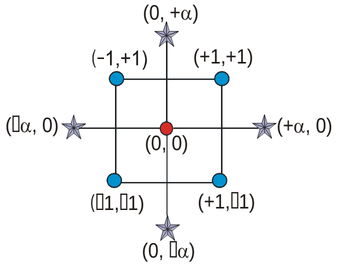
____

Choose a design: Type:____

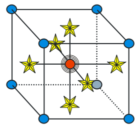
Replicates: ____, Blocks: _____, Center points: ____

RSM Design Selection

Central Composite Designs (CCD):



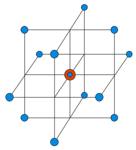
- ☐ **Standard** (axial levels (α) for “star points” are set for rotatability): Good design properties, little collinearity, rotatable, orthogonal blocks, insensitive to outliers and missing data. Each factor has five levels. Region of operability must be greater than region of interest to accommodate axial runs. <u>For 5 or more factors, change factorial core of CCD to</u>:
  - Standard Resolution V **fractional** design, or
  - **Min-run Res V**.



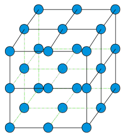
- ☐ **Face-centered** **(FCD)** (α = 1.0): Each factor conveniently has only three levels. Use when region of interest and region of operability are nearly the same. Good design properties for designs up to 5 factors: little collinearity, cuboidal rather than rotatable, insensitive to outliers and missing data. (<u>Not recommended for six or more factors</u> due to high collinearity in squared terms.)

- ☐ **Practical alpha** (α = 4th-root of k – the number of factors): Recommended for six or more factors to reduce collinearity in CCD.

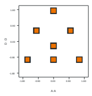
- ☐ **Small (Draper-Lin)** A minimal design <u>not recommended</u> being very sensitive to bad data.



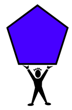
Box-Behnken (BBD): Each factor has only three levels. Good design properties, little collinearity, rotatable or nearly rotatable, some have orthogonal blocks, insensitive to outliers and missing data. Does not predict well at the corners of the design space. Use when region of interest and region of operability nearly the same.



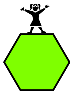
3-Level Factorial: Good for three factors at most. Beyond that the number of runs far exceeds what’s needed for a good RSM. (*See table on next page - Number of Design Points for Various RSM Designs*). Good design properties, cuboidal rather than rotatable, insensitive to outliers and missing data. To reduce runs for more than three factors, consider BBD or FCD.



Hybrid: This “Legacy” design is <u>not recommended</u> due to being very sensitive to bad data (but better than the Small CCD). Runs are oddly spaced as shown in the figure) with each factor having four or five levels. Region of operability must be greater than region of interest to accommodate axial runs.



Pentagonal: This is a “Legacy” design for <u>two factors only</u> which provides an interesting geometry with one apex (1, 0) and 4 levels of one factor versus 5 of the other. It may be of interest with one categoric factor at two levels to form a three-dimensional region with pentagonal faces on the two numeric (RSM) factors.



Hexagonal: Another “Legacy” design for <u>two factors only</u>, which provides a good alternative to the pentagon with 5 levels of one factor versus 3 of the other.

Optimal (custom): Handles any or all input types, e.g., numeric discrete and/or categoric, within any constraints for the bare minimum of runs required for the specified polynomial model. By default, Stat-Ease software adds optional "Lack of fit" (LOF) points (check runs) via distance-based criteria and "Replicate" points for pure error estimation. They bolster the design and provide for LOF testing.

Choose from these optimal criteria:

- I (default) - reduces the average prediction variance. (Best predictions)
- D - minimizes the joint confidence interval for the model coefficients. (Best for finding effects, so this becomes the default for factorial designs)
- A - minimizes the average confidence interval.
- Distance based - not recommended: chooses points as far away from each other as possible, thus achieving maximum spread.

and the following exchange algorithms:

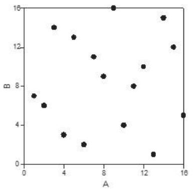
- Best (default) - chooses the best from Point or Coordinate exchange.*
  **See the Summary after building the design to see which method won out.*
- Point exchange – based on geometric candidate set, coordinates fixed*.
  **Often produces levels that round better for easier setting.*
- Coordinate exchange – candidate-set free: Points located anywhere*.
  **Generally provides more optimal levels but often not rounded very well.*

Definitive Screen (DSD): A "Supersaturated" three-level design for RSM which aliases squared terms with two-factor interactions (2FI). These designs are useful for screening main effects. They may reveal information about the second-order model terms and thus be helpful for optimization, but this requires a lot of assumptions, perhaps too many: Proceed with caution.

Split-Plot Central Composite (SPCCD): Handles hard-to-change (HTC) factors using a standard RSM template. For more than a few factors the SPCCD may generate more runs than needed for proper design sizing. If so, go to the Optimal alternative for split-plot RSM.

Split-Plot Optimal (custom): Good choice when one of more factors are HTC (generally better than SPCCD) and only option when factors are discrete and/or categoric or when constraints form an irregular experimental region.

Latin Hypercube: This "Space-Filling" design (available in Stat-Ease 360) stratifies factors uniformly throughout the experimental region by dividing each into the same number of equally spaced intervals, each with exactly one setting. By not producing replicates, it is a good choice for experiments on simulations that produce deterministic results.



Optimal Distance: Another "Space-Filling" design (also available in Stat-Ease 360) that, based on distance or by your chosen model, handles irregularly shaped regions.

Number of Points for Standard RSM Designs

Number Factors	CCD full	CCD fractional	CCD MR-5	Box-Behnken	Small CCD*	DSD**	Quadratic Coefficients†
2	13	NA	NA	NA	NA	NA	6
3	20	NA	NA	17	15	NA	10
4	30	NA	NA	29	21	13	15
5	50	32	NA	46	26	13	21
6	86	52	40	54	33	13	28
7	152	88	50	62	41	17	36
8	272	154 90	60	120	51	17	45
9	540	284 156	70	130	61	21	55
10	X	286 158	82	170	71	21	66
20	X	562	258	348	X	44	231
30	X	X	532	X	X	61	496
40	X	X	908	X	X	NA	901
50	X	X	1382	X	X	NA	1376

X = Excessive runs

NA = Not Available

* Small central composite design (Draper-Lin) not recommended—too minimal.

** DSDs lack enough runs to estimate all terms in the quadratic model. Not advised/

† Including the intercept, linear, two-factor interaction, and quadratic (squared) terms.

Mixture Design Worksheet

Identify opportunity and define objective:____

____

____

State objective in terms of measurable responses:

- *Define the precision (d - signal) required for each response.*
- *Estimate the experimental error (σ - noise) for each response.*
- *Use d/σ (signal to noise) to check for adequate precision using FDS.*

Name	Units	d	Σ	FDS	Goal
R_1:					
R_2:					
R_3:					
R_4:					

Select the components and ranges to vary within the experiment:

Name	Units	Type	Low	High
A:				
B:				
C:				
D:				
E:				
F:				
G:				
		Mix Total:		

Quantify any MultiLinear Constraints (MLC's):

____

____

Choose a design: Type:_______________________________________

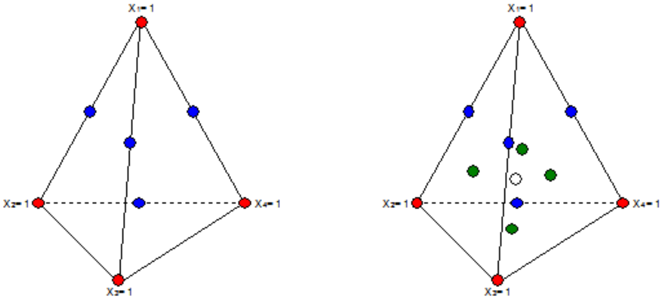
Replicates: ____, Blocks: _____, Centroids: ____

Mixture Design Selection

Simplex: Applicable if all components range from 0 to 100 percent (no constraints) or they have same range (necessary, but not sufficient, to form a simplex geometry for the experimental region).

- ☐ **Lattice**: Specify degree “m” of polynomial (1 - linear, 2 - quadratic or 3 - cubic). Design is then constructed of m+1 equally spaced values from 0 to 1 (coded levels of individual mixture component). The resulting number of blends depends on both the number of components (“q”) and the degree of the polynomial “m”. Centroid not necessarily part of design.

- ☐ **Centroid**: This is a “Legacy” design that always includes the centroid always with the 2^q-1 distinct mixtures generated from permutations of:
  - ○ Pure components: (1, 0, ..., 0)
  - ○ Binary (two-part) blends: (1/2, 1/2, 0, ..., 0)
  - ○ Tertiary (three-part) blends: (1/3, 1/3, 1/3, 0, ..., 0)
  - ○ and so on to the overall centroid: (1/q, 1/q, ..., 1/q)



Simplex Lattice versus Simplex Centroid

Screening: Essential for six or more components. Creates design for linear model only to find the components with strong main effects.

- ☐ **Simplex**
- ☐ **Extreme vertices** (for non-simplex—detected automatically)

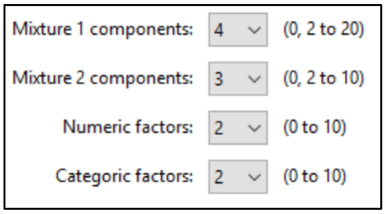
Custom:

- ☐ **Optimal**: *(See RSM section for details.)* Required for non-simplex (detected automatically) or you want to fit a tailored model; e.g., reduced quadratic.

Custom Design Selection

Optimal (Combined): These designs combine mixture components with numerical (including amounts) and/or categoric process factors. For example, if you want to make filled cupcakes and bake them in two ovens, set up:

- ☐ Mixture 1 components – the cake: 4 for flour, water, sugar and eggs
- ☐ Mixture 2 components – the filling: 3 for cream cheese, salt and chocolate
- ☐ Numeric factors – the baking process: 2 for time and temperature
- ☐ Categoric factors – the oven: 2 types – Electric or gas.



Mixture 1 components: 4 (0, 2 to 20)
Mixture 2 components: 3 (0, 2 to 10)
Numeric factors: 2 (0 to 10)
Categoric factors: 2 (0 to 10)

Split plots come in handy for combined designs such as this. For example, consider mixing up muffins individually by random order (being easy to change) and then baking a group of them together at specified times and temperatures—accomplished by designating these process factors as hard to change.

Default crossed models often require excessive run. To reduce design size, via "Edit model" on the Optimal specification screen, consider:

- Selecting the KCV model, which only crosses terms for the primary mixture-process interaction—a good compromise, or
- Restricting the "Combined Order Limit" to Cubic.

See RSM section for more details on optimal design options and defaults.

Note: The model for categoric factors takes the same order as for the numeric ones (process). For example, if quadratic (default), the second-order two-factor interaction (2FI) model will be selected for the categoric factors.

Existing: Allows for import of already-collected results either directly ("**Import Data Set**"), or via a dummy design ("**Blank Spreadsheet**"). Be careful: Models from happenstance data may be useless for prediction. Tips:

- Runs must have all factor levels specified
- Missing response values should be left blank, not entered as zeroes
- Categoric levels must be consistently spelled out in the actual data and design (capitalization sensitive)
- Set low (-1) and high (+1) factor levels to the actual minimum and maximum.
- Use Graph Columns to plot each factor versus all other factors. Watch out for:
    - Outlying points that create high leverage on the resulting model. *(If removed, re-set the -1/+1 coding to get a better analysis.)*
    - High correlations with very bright blue or red colors on the matrix.
- Use Evaluation to assess the quality of the input data for variance-inflation factors (VIF's), leverages, etc.—see "Design Evaluation" guide for details
- In the Analysis, use AutoSelect tools to reduce the model—see guide on "Automatic Model Selection" for details
- Consider following up with a designed experiment based on what you learned.

Simple Sample: Use this design choice as a tool for entering raw data to generate basic statistics (mean, standard deviations and intervals) for a process where no inputs are intentionally varied. There are no factors to enter—only a specified number of observations (runs containing one or more measured responses).

User-Defined: Generates points based on geometry of design space.

Design Evaluation Guide

1. Select the polynomial model you want to evaluate. First look for aliases. No aliases should be found. If the model is aliased, the program calculates the alias structure -- examine this. An aliased model implies there are either not enough unique design points, or the wrong set of design points was chosen.

2. Examine the table of degrees of freedom (df) for the model. You want:
   a) Minimum 3 lack-of-fit df.
   b) Minimum 4 df for pure error.
3. Look at the standard errors (based on s = 1) of the coefficients. They should be the same within type of coefficient. For example, the standard errors associated with all the linear (first order) coefficients should be equal. The standard errors for the cross products (second order terms) may be different from those for the linear standard errors, but they should all be equal to each other, and so on.
4. Examine the variance inflation factors (VIF) of the coefficients:

$$\text{VIF} = \frac{1}{1-R_i^2}$$

☐ VIF measures how much the lack of orthogonality in the design inflates the variance of that model coefficient. (Specifically, the standard error of a model coefficient is increased by a factor equal to the square root of the VIF, when compared to the standard error for the same model coefficient in an orthogonal design.)

☐ VIF of 1 is ideal because then the coefficient is orthogonal to the remaining model terms, that is, the correlation coefficient (R_i^2) is 0.

☐ VIFs above 10 are cause for concern.

☐ VIFs above 100 are cause for alarm, indicating coefficients are poorly estimated due to multicollinearity.

☐ VIFs over 1000 are caused by extreme collinearity.

☐ VIF's are not an appropriate measure for mixture designs, which are inherently non-orthogonal.

5. For factorial designs: Look at the power calculations to determine if the design is likely to detect the effects of interest. Degrees of freedom for residual error must be available to calculate power, so for unreplicated factorial designs, specify main effects model only. For more details, see **Power Calculation Guide**.

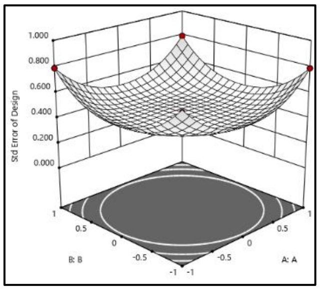
   For RSM and mixture designs: look at fraction of design space (FDS) graph to evaluate precision rather than power. (**See FDS Guide**.)

6. Examine the leverages of the design points. Consider replicating points where leverage is more than 2 times the average and/or points having leverage approaching 1.

$$\text{Average leverage} = \frac{p}{N}$$

Where “p” is the number of model terms including the intercept (and any block coefficients) and “N” is the number of experiments.

7. Go to **Graphs, Contour** (or 3D Surface) Do a plot of the standard error (based on s = 1). The shape of this plot depends only on the design points and the polynomial being fit. Ideally the design produces a flat error profile centered in the middle of your design space. For an RSM design this should appear as either a circle or a square of uniform precision.



Repeat the “design evaluation – design modification” cycle until satisfied with the results. Then go ahead and run the experiment.

Matrix Measures (for more thorough evaluation by statistical researchers)

1. Evaluate measures of your design matrix:
    a. Condition Number of Coefficient Matrix (ratio of max to min eigenvalues, or roots, of the X'X matrix):

        $\kappa = \lambda_{max}/\lambda_{min}$

• $\kappa = 1$	no multicollinearity, i.e., orthogonal
• $\kappa < 100$	multicollinearity not serious
• $\kappa < 1000$	moderate to strong multicollinearity
• $\kappa > 1000$	severe multicollinearity

        {Note: Since mixture designs can never be orthogonal, the matrix condition number cannot be evaluated on an absolute scale.}
    b. Maximum, Average, and Minimum mean prediction variance of the design points. These are estimated by the Fraction of Design Space sample. They are the variance multipliers for the prediction interval around the mean.
    c. G Efficiency – this is a simple measure of average prediction variance as a percentage of the maximum prediction variance. If possible, try to get a G efficiency of at least 50%. *Note: Lack-of-fit and replicates tend to reduce the G efficiency of a design.*
    d. Scaled D-optimality – this matrix-based measure assesses a design's support of a model in terms of prediction capability. It is a single-minded criterion which often does not give a true measure of design quality. To get a more balanced assessment, look at all the measures presented during design evaluation. The D-optimality criterion minimizes the variance associated with the coefficients in the model. When scaled the formula becomes:

        $$N((\text{determinant of }(X'X)^{-1})^{1/p})$$

        Where N is the number of experiments and p is the number of model terms including the intercept and any block coefficients. Scaling allows comparison of designs with different number of runs. The smaller the scaled D-optimal criterion the smaller the volume of the joint confidence interval of the model coefficients.
    e. The determinant, generalized equivalence condition, trace and I-score are relative measures (the smaller the better!) used to compare designs having the same number of runs, primarily for algorithmic point selection. It is usually not possible to minimize all three simultaneously.
        - The determinant (related to D-optimal) measures the volume of the joint confidence interval of the model coefficients.
        - The trace (related to A-optimal) represents the average variance of the model coefficients.
        - The I-score (related to I-optimal) measures the integral of the prediction variance across the design space.
2. Examine the correlation matrix of the model coefficients (derived from $(X'X)^{-1}$). In an orthogonal design all correlations with other coefficients are zero. How close is your design to this ideal?

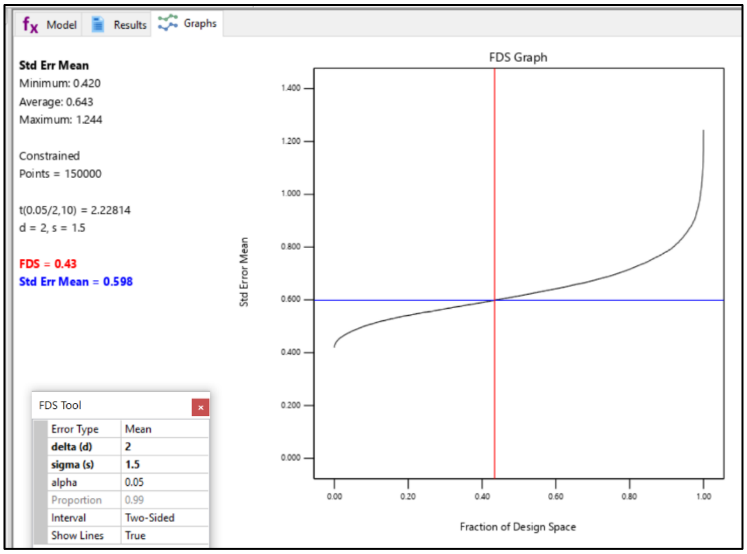
    {Note: Due to the constraint that the components sum to a constant, mixture designs can never be orthogonal.}
3. Examine the correlation matrix of the independent factors (comes directly from the X matrix itself). In an orthogonal design none of the factors are correlated. *Mixture designs can never be orthogonal.*

4. Modify your design based on knowledge gained from the evaluation:
    a. Add additional runs manually or via the design tools in Stat-Ease software for augmenting any existing set of runs.
    b. Choose a different design.

Fraction of Design Space (FDS) Guide

FDS calculations help experimenters properly size response surface (RSM) and mixture designs, for which the normal power calculations lose relevance. At a minimum (going with defaults), supply the delta (d)—the desired precision or difference to be detected—and the sigma (s)—the “noise” measured by standard deviation from process data. The FDS graph will then reveal the fraction of the design region that can predict the response with sufficient precision: 0.8 (80 percent) or more, ideally.

FDS results depend on the model specified the initial tab in Evaluation as well as the following options:



```
Std Err Mean
Minimum: 0.420
Average: 0.643
Maximum: 1.244

Constrained
Points = 150000

t(0.05/2,10) = 2.22814
d = 2, s = 1.5

FDS = 0.43
Std Err Mean = 0.598
```

Error Type	Mean
delta (d)	**2**
sigma (s)	**1.5**
alpha	0.05
Proportion	0.99
Interval	Two-Sided
Show Lines	True

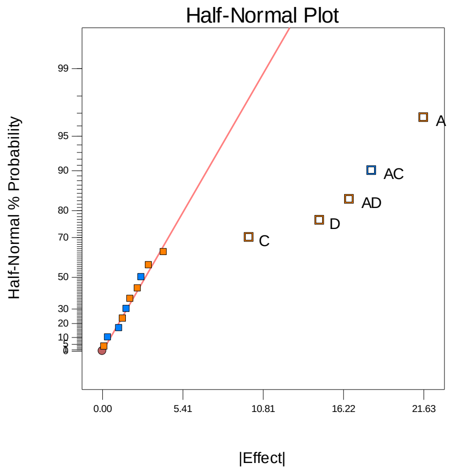
- **Error Type**:
    - **Mean** (default) – best for typical process and/or product optimization
    - **Pred** – choose when the focus is the individual outcomes. Note: More runs are required to get similar precision with “Pred” than “Mean”.
    - **Diff** – recommended when searching for any change in the response, such as for verification. Smaller changes are more difficult to detect.
    - **Tolerance** – useful for setting specifications
- **delta (d)** – (required entry) difference in response deemed important—this change being defined by Error Type:
    - Mean: half-width of confidence interval (CI), that is, precision of prediction
    - Pred: half-width of prediction interval (PI)
    - Tolerance: half-width of the tolerance interval (TI)*
      **Must specify Proportion of population (default 0.99)*
    - Diff: minimum change in the response that is important to detect.
- **sigma (s)** – standard deviation (default 1—override with correct value)
- **alpha** – significance level (default 0.05).
- **proportion** – grayed out if Error Type not set at Tolerance
- **Interval** – One-Sided or Two-Sided (default)
- **Show lines** – Toggles crosshairs off when changed from True (default) to False*
  **If delta to sigma ratio exceed a certain level beyond FDS of 1, no crosshairs show.*

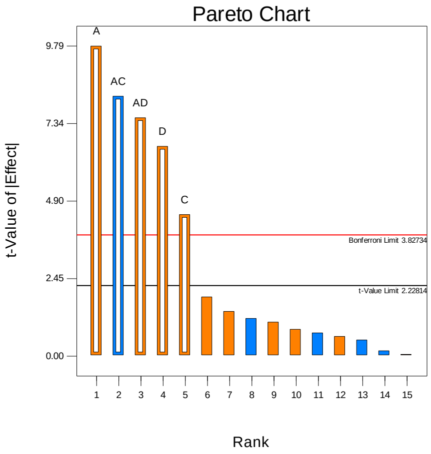
Section 2:

Analyzing the Results

Factorial Analysis Guide

1. Compute effects. Use half-normal plot to select model. Click the biggest effect (point furthest to the right) and continue right-to-left until the line runs through points nearest zero (or lasso all of them at once via a mouse drag). Alternatively (*but not as easy to do*), on the Pareto Chart pick effects from left to right, largest to smallest, until all other effects fall below the Bonferroni and/or t-value limit. Best approach: Pick effects using half-normal and then view them on Pareto as a 'double check'.





2. Choose ANOVA* (Analysis of Variance) and check the selected model:
   **(For split plots via REML—Restricted Maximum Likelihood—with p-values fine-tuned via Kenward-Roger method.)*

   a) Review the ANOVA results.

      ☐ P-value < 0.05: significant.

      ☐ P-value > 0.10: not significant.

   b) Examine the F tests on the regression coefficients. Look for terms that can be eliminated, i.e., terms having (Prob > F) > 0.10. Be sure to maintain hierarchy as advised by the software.

   c) Examine the F tests for the lack of fit (available only with measures of pure error from replicated runs). If insignificant, continue with the analysis. If lack of fit tests significant, look at the graphs to determine if a more complex model is necessary. If the model is useful as is, use it.

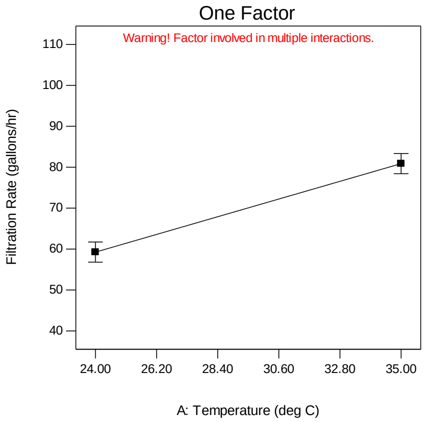
   d) Check for "Adeq Precision" > 4. This is a signal to noise ratio (see formula in Response Surface Analysis Guide).
      (Not available for split plots.)

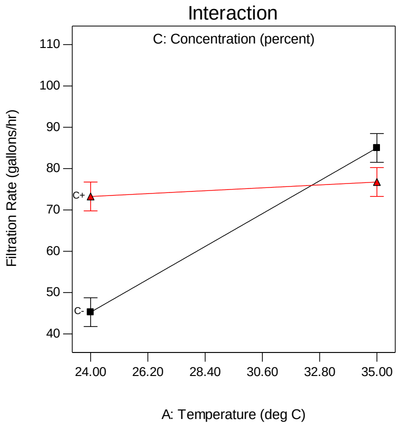
3. Refer to the **Residual Analysis and Diagnostic Plots Guide**.
   Verify the ANOVA assumptions by looking at the residual plots.

4. Explore the region of interest:

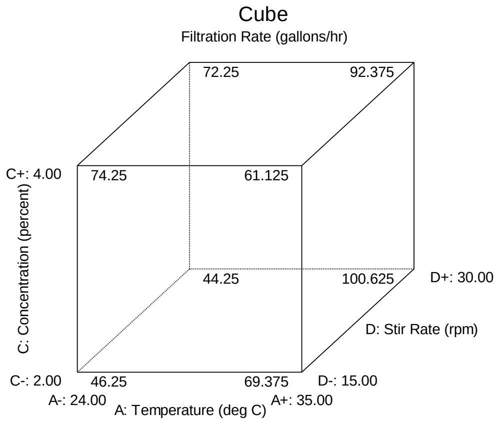
a) One Factor plot (don't use for factors involved in interactions):

b) Interaction plot (with 95% Least Significant Difference (LSD) bars):

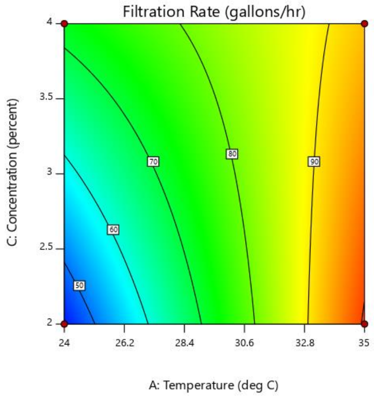


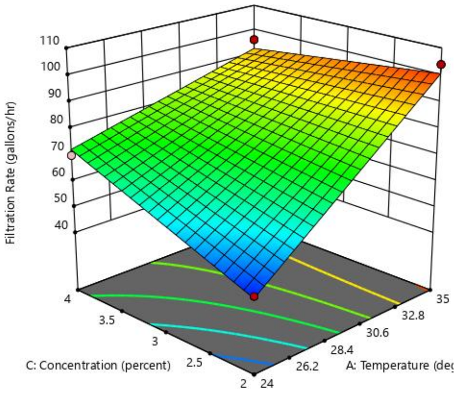


c) Cube plot (especially useful if three factors are significant):



d) Contour plot and 3D surface plot:





Response Surface/Mixture Analysis Guide

1. Select a model *(skip this step for split plots)*:
    a) Note which models are aliased: These <u>must not be selected</u>.
    b) *"Fit Summary*": Focus on the "Suggested" model.
    c) "*Sequential Model Sum of Squares*": Select the highest order polynomial where the additional terms are significant and the model is not aliased.

        p-value < 0.05 ☺ p-value > 0.10 ☹

    d) "*Model Summary Statistics*": Focus on the model with high *"Adjusted R-Squared"* and high *"Predicted R-Squared".*
    e) "*Lack of Fit (LOF) Tests*": Selected model should exhibit insignificant LOF.

        p-value < 0.05 ☹ p-value > 0.10 ☺

        No lack of fit reported? If so, the design lacks:
        i. Excess unique points beyond the number of model terms (to estimate variation about fitted surface), and/or
        ii. Replicate runs to estimate pure error (needed to statistically assess the lack of fit).

2. Check the selected model:
    a) Review the ANOVA *(for split plots use REML—Restricted Maximum Likelihood—with p-values fine-tuned via Kenward-Roger method)*. The F-test is for the complete model, rather than just the additional terms for that order model as in the sequential table. Model should be significant (p-value < 0.05) and lack-of-fit insignificant (P-value > 0.10).
    b) Examine the F tests on the regression coefficients - can the complexity of the polynomial be reduced? Look for terms that can be eliminated, i.e., coefficients having p-values > 0.10. Be sure to maintain hierarchy. If there are many such terms, consider model reduction via **Auto Select** (for guidance see *Handbook* section on "Automatic Model Selection").
    c) Check for "*Adeq Precision*" > 4 *(not available for split plots)*. This is a signal to noise ratio given by the following formula:

$$\left[\frac{\max\left(\hat{Y}\right)-\min\left(\hat{Y}\right)}{\sqrt{\bar{V}\left(\hat{Y}\right)}}\right]>4 \qquad \bar{V}\left(\hat{Y}\right)=\frac{1}{n}\sum_{i=1}^{n}V\left(\hat{Y}\right)=\frac{p\sigma^{2}}{n}$$

        p = number of model parameters (including intercept (b_0) and any block coefficients)

        σ^2 = residual MS from ANOVA table

        n = number of experiments

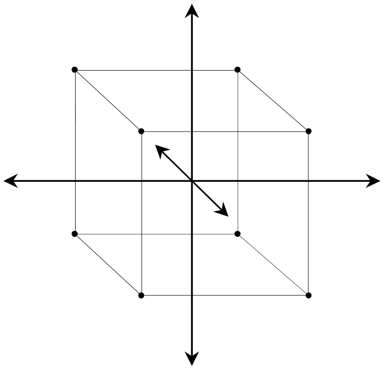
    d) Check that "*Pred R-Squared*" *(not available for split plots)* falls no more than 0.2 below the "*Adj R-Squared*". If so, consider model reduction.

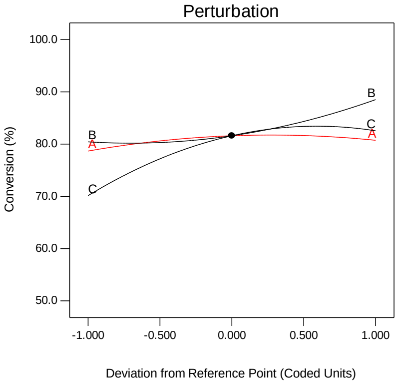
3. Refer to the **Residual Analysis and Diagnostics Plots Guide**. Verify the ANOVA assumptions by looking at the residual plots.

4. Explore the region of interest:

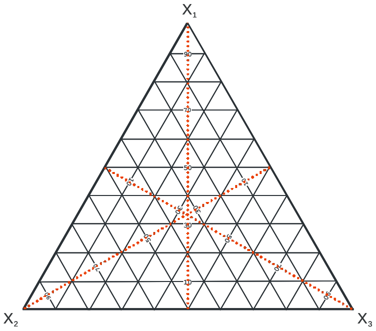
   a) Perturbation/Trace plots to choose the factor(s)/component(s) to “slice” through the design space. Choose ones having small effects (flat response curve) or components having linear effects (straight). In the RSM and mixture examples below, take slices of factor “A”.

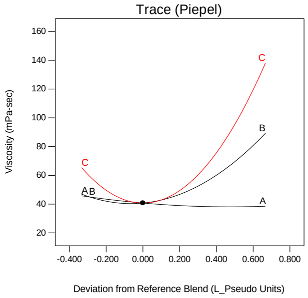
   - <u>RSM perturbation</u> plot





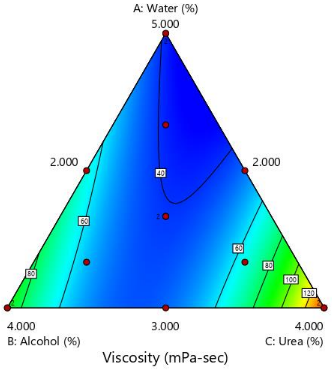
   - <u>Mixture trace</u> plot *(view Piepel’s direction for broadest paths)*

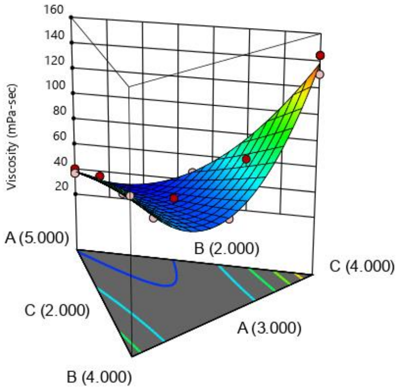




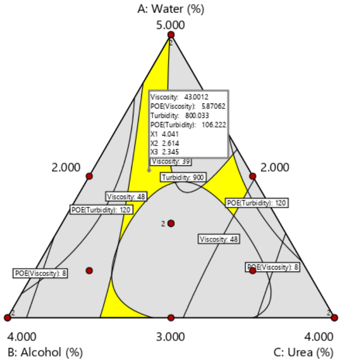
Perturbation/Trace plots are particularly useful after finding optimal points. They show how sensitive the optimum is to changes in each factor or component.

   b) Contour plots (shown below for a mixture design) to explore your design space, slicing on the factors/components identified from the perturbation/trace plots as well as any categorical factors.





5. Perform “Numerical” optimization to identify most desirable factor (component) levels for single or multiple responses. View the feasible window (‘sweet spot’) via “Graphical” optimization (‘overlay’ plot). *See **Optimization Guide** for details.*



6. See the Confirmation node under the Post Analysis branch for the prediction interval (PI) expected for individual confirmation runs. Perform a number—six is good—of confirmation runs, enter them in to generate their mean in comparison to the PI recalculated for the sample size. Ideally it will fall within range. ☺ If not, consider what may have changed between the time you did the experiment and the subsequent confirmation runs. ☹

Combined Mixture/Process Analysis Guide

1. Select a model *(skip this step for split plots)*: Look for what is suggested in the Fit Summary table, in this case: quadratic mixture by linear process (QxL). Often, as in this case, it is the one with the highest adjusted and predicted R-squared (row [12] 0.9601 and 0.9240).

Combined Model Fit Summary Table

	Mixture Order	Process Order	Mixture p-value	Process p-value	Adjusted R^2	Predicted R^2	
1	M	M					
2	M	L		< 0.0001	0.3916	0.3329	
3	M	2FI		0.9883	0.3561	0.2507	
4	M	Q	*	*	0.3561	0.2507	*Aliased
5	M	C	*	0.8488	0.3432	0.2198	*Aliased
6	L	M	< 0.0001		0.4460	0.3919	
7	L	L	< 0.0001	< 0.0001	0.9211	0.8889	
8	L	2FI	< 0.0001	0.6237	0.9177	0.8341	
9	L	Q	< 0.0001		0.9177	0.8341	Aliased
10	L	C	< 0.0001	0.9176	0.9113	0.7729	Aliased
11	Q	M	0.5401		0.4373	0.3487	
12	**Q**	**L**	**0.0003**	**< 0.0001**	**0.9601**	**0.9240**	**Suggested**
13	Q	2FI	0.0028	0.1129	0.9736	0.8796	
14	Q	Q	0.0028	*	0.9736	0.8796	*Aliased
15	Q	C	0.0389	0.7158	0.9683	-0.9445	Aliased
16	SC	M	0.6426		0.4284	0.3348	
17	SC	L	0.4611	< 0.0001	0.9598	0.9181	
18	SC	2FI	0.2567	0.1212	0.9802	0.8391	
19	SC	Q	0.2567	*	0.9802	0.8391	*Aliased
20	SC	C	*	*			*Aliased

This sequential table shows the significance of terms added layer-by-layer to the model above. For example, in this case starting with the mean by mean (MxM) model (row [1]):

- Linear (L) process terms provide significant information beyond the mean (M) model ($p<0.0001$ in row [2]).
- Adding 2FI process terms provides no benefit ([3] $p=0.9883$).
- The next two models (rows [4] and [5]), MxQ and MxC are aliased – do not pick them!
- Start again from MxM.
- Linear (L) mixture terms are significant ([6] $p<0.0001$).
- L process terms are a significant addition ([7] $p<0.0001$).
- 2FI process terms do not add significantly ([8] $p=0.6237$).
- The next two models ([9] and [10]), LxQ and LxC are aliased.
- Adding the Q-Mix terms provides no benefit ([11] $p=0.5401$).
- Add L process terms to significantly improve the model fit ([12] $p<0.0001$).
- Adding the 2FI terms provide little benefit ([13] $p=0.1129$) and they reduce the predicted R^2.

Thus, the QxL combined mixture-by-process model is suggested.

2. Due to the complexity of combined models, try **Automatic Model Selection** to remove unnecessary terms from the model.

See the next page for the details of automatic model selection.

Automatic Model Selection

Automatic model selection (“Auto Select”) algorithmically chooses the terms to best keep in the model—discarding the others. Four criteria are provided: AICc, BIC, p-value, and Adjusted R-Squared with four selection methods: Forward, Backward, Stepwise and All Hierarchical. The table below shows the eight available combinations.

	Selection			
Criterion	**Forward**	**Backward**	**Stepwise**	**All Hierarchical**
AICc	Yes*	Yes*	No	No
BIC	Yes*	Yes*	No	No
p-value	Yes	Yes*	Yes	No
Adjusted R-Squared	No	No	No	Yes

* *Best selection method for the given criterion*

Automatic Model Selection cannot substitute for your judgment based on subject-matter knowledge. Please take the time to review the results on the ANOVA and Diagnostics before using the analysis to make decisions.

You are encouraged to use multiple combinations of the criterion and selection directions to help decide which terms form the best model. AICc with forward selection is the default and best general method for selecting the model. We suggest you also try a backward AICc selection. P-value using backward selection is also recommended and may be more familiar.

Details on Criterion:

- “AICc “stands for Akaike Information Criterion corrected for a small design. *Akaike* is pronounced (ah kah ee Kay). It measures goodness of fit.
- “BIC” stands for Bayesian Information Criterion. It is an alternative to AICc that may work better for larger designs and models.
- “p-value” is the standard method looking for significant terms to keep and/or insignificant terms to remove from the model.
- “Adjusted R-squared” relates how well the current model explains the data after an adjustment (from raw R-squared) to prevent too many terms.

Details on Selection:

- “Forward” adds terms to a model that improve the criterion.
- “Backward” removes terms from a model that are detrimental to the criterion.
- “Stepwise” is a combination of forward and backward that first adds terms that improve the criterion, then checks whether any terms should be removed.
- “All hierarchical” selection checks all possible models that maintain hierarchy, keeping the one with the best criterion score.

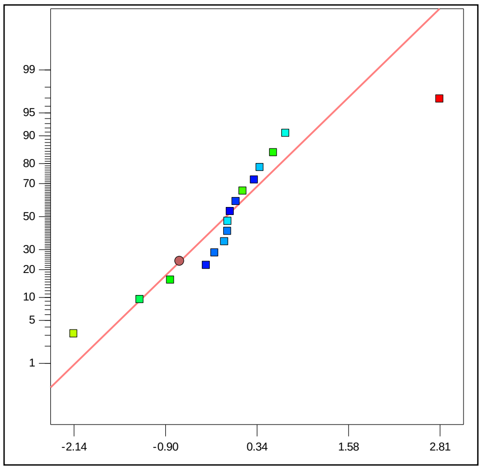
Residual Analysis and Diagnostic Plots Guide

Residual analysis is necessary to confirm that the assumptions for the ANOVA are met. Other diagnostic plots may provide interesting information in some situations. ALWAYS review these plots!

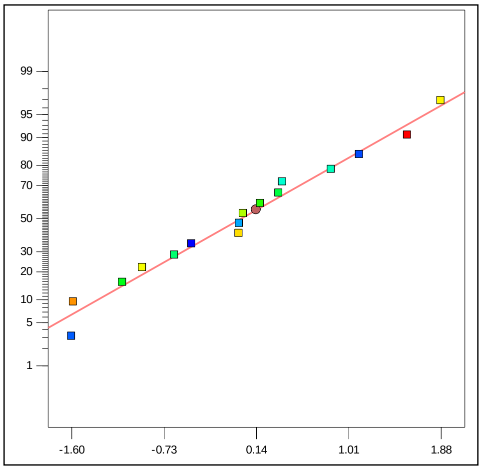
A. **Diagnostic plots**

1. Plot the (externally) <u>studentized residuals</u>:

   a) **Normal plot** - should be straight line.

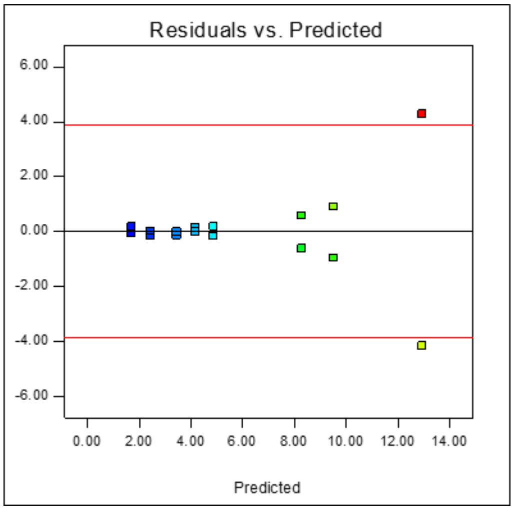


BAD: S shape

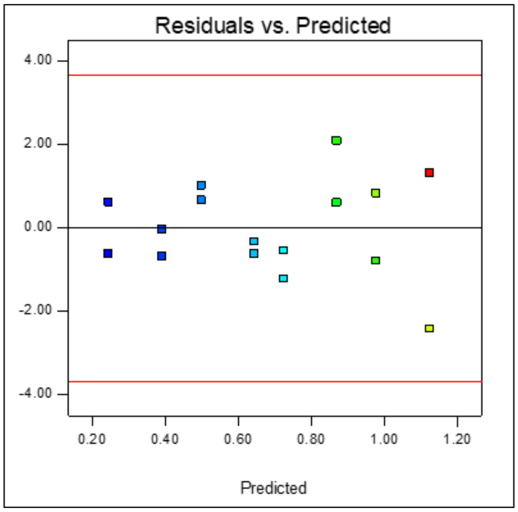


GOOD: Linear or Normal

   b) **Residuals (e_i) vs predicted** - should be random scatter.

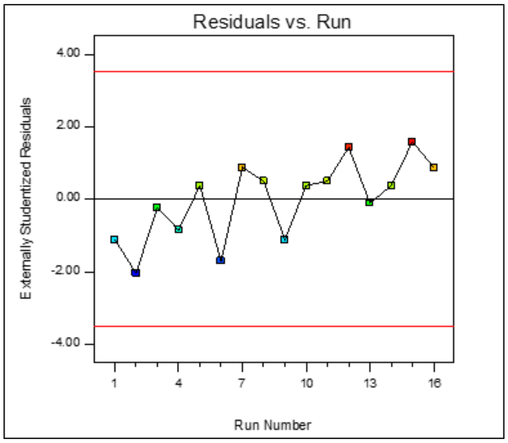


BAD: Megaphone shape

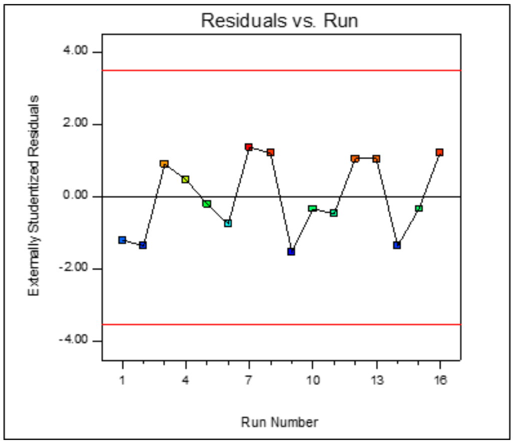


GOOD: Random scatter

c) **Residuals (e_i) vs run** - should be random scatter, no trends.



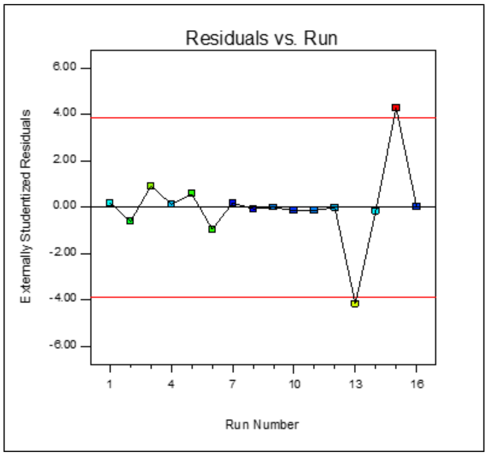
BAD: Trend



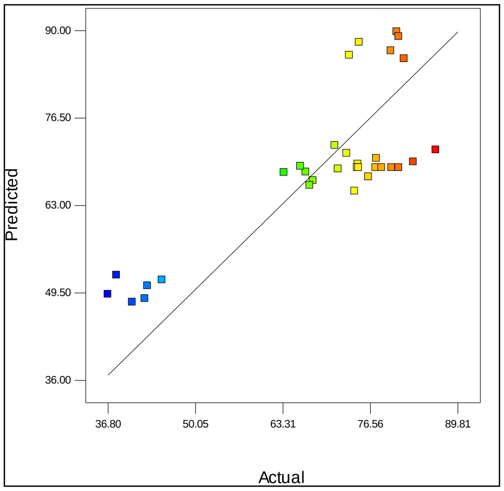
GOOD: No pattern

Also, look for **externally studentized residuals** outside limits. These runs are statistical outliers that may indicate:

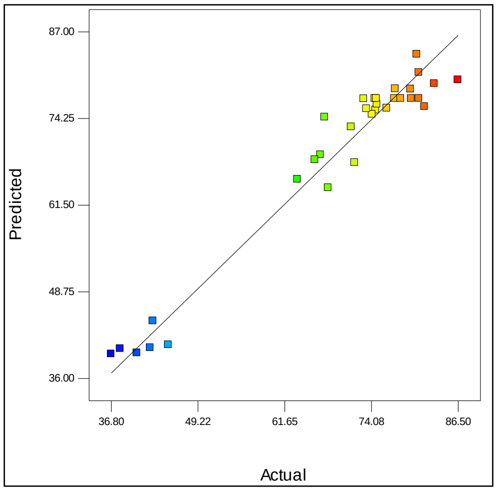
- a problem with the model,
- a transformation,
- a special cause that merits ignoring the result or run.



2. View the **predicted vs actual** plot whose points should be randomly scattered along the 45-degree line. Groups of points above or below the line indicate areas of over or under prediction.

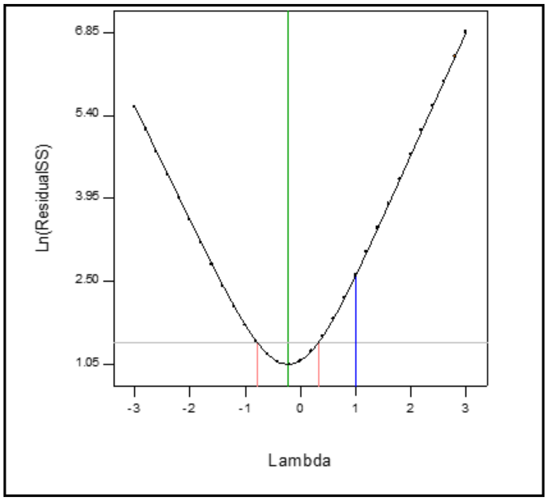


Poor Prediction

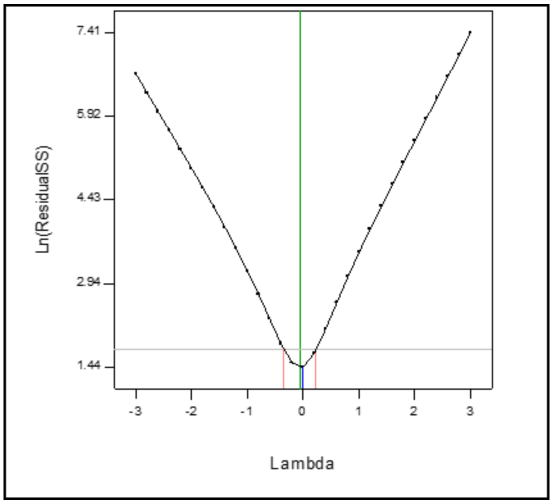


Better Prediction

3. Use the **Box-Cox plot** to determine if a power law transformation might be appropriate for your data. The blue line indicates the current transformation (at Lambda =1 for none) and the green line indicates the best lambda value. Red lines indicate a 95% confidence interval associated with the best lambda value. Stat-Ease software recommends the standard transformation, such as log, closest to the best lambda value unless the confidence interval includes 1, in which case the recommendation will be "None."

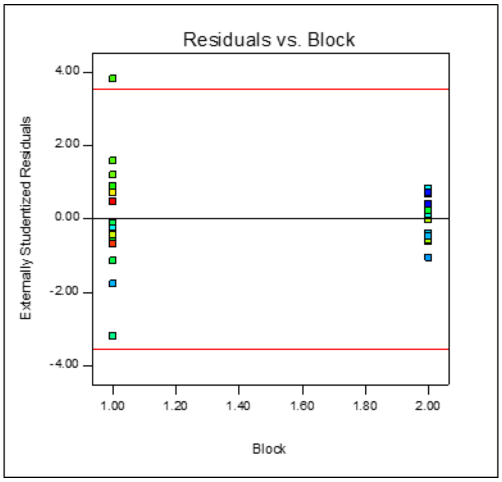


Before Transformation

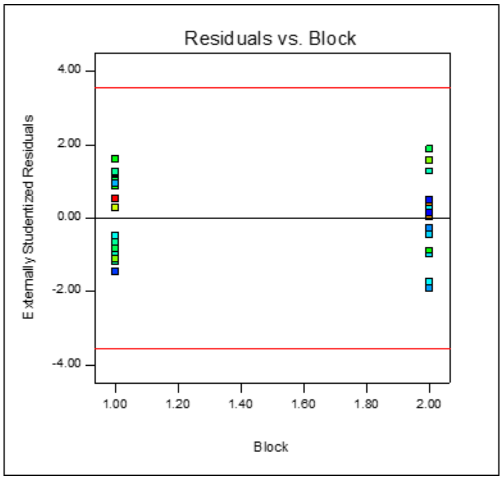


After Transformation

4. **Residuals (e_i) vs factor** – especially useful with blocks. Should be split by the zero-line at either end of the range – no obvious main effect (up or down). If you see an effect, go back, add it to the predictive model and assess its statistical significance. Relatively similar variation between levels. Watch ONLY for very large differences.



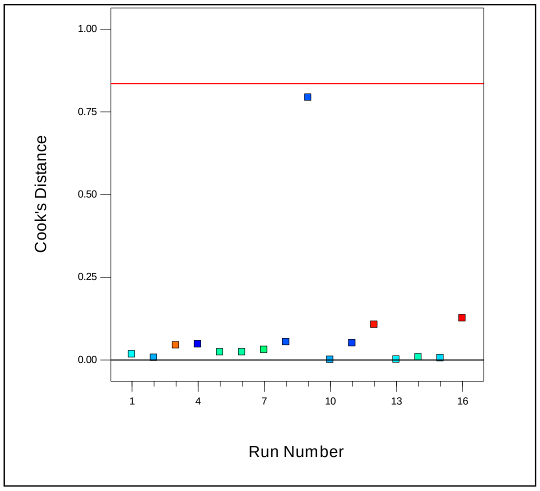
BAD: More variation at one end



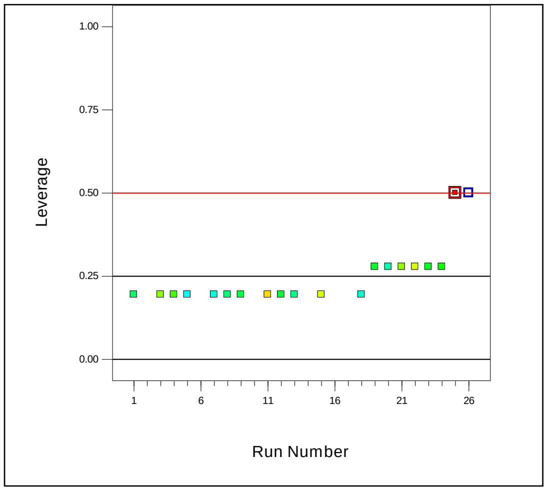
GOOD: Random scatter both ends

Influence plots – review if Diagnostic plots indicate problems

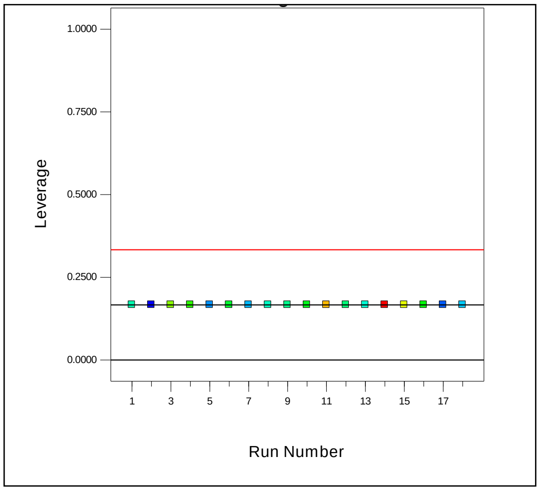
1. **Cook's Distance** helps if you see more than one outlier in other diagnostic plots. Investigate the run with the largest Cook's Distance first. Often, if this run is ignored due to a special cause, other apparent outliers can be explained by the model.



2. Watch for **leverage vs run** values at or beyond twice the average leverage. These runs will unduly influence at least one model parameter. If identified prior to running the experiment, it can be replicated to reduce leverage. Otherwise, all you can do is check the actual responses to be sure they are as expected for the factor settings. Be especially careful of any leverages at one (1.0). These runs will be fitted exactly with no residual!



BAD: Some at twice the average

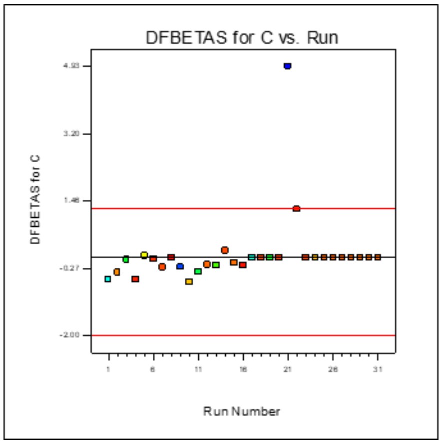


GOOD: All the same

3. <u>Deletion diagnostics</u> – statistics calculated by taking each run out, one after the other, and seeing how this affects the model fit.

   a) **DFFITS** (difference in fits) is another statistic helpful for detecting influential runs. Do not be overly alarmed at points outside of limits: Just check that they are not extraordinary. If earlier diagnostics show outliers that do not go out of bounds on DFFITS, then these do not create a significant difference in fits and thus you need not be overly concerned. ☺

b) **DFBETAS** (difference in beta coefficients) breaks down the impact of any given run on a particular model term. If you see an excessive value, consider whether a factor in the term falls beyond a reasonable range (for example, it may be that an axial (star) point in a CCD projects outside of the feasible operating region) and, if so, try ignoring this particular run.



Statistical Details on Diagnostic Measures

Residual ($e_i = y_i - \hat{y}_i$):

Difference between the actual individual value (y_i) and the value predicted from the model ($\hat{y}_i$).

Leverage ($h_{ii} = x_i^T \left(X^T X\right)^{-1} x_i$ *where x is factor level and X is design matrix*):
Numerical value between 0 and 1 that indicates the potential for a case to influence the model fit. A leverage of 1 means the predicted value at that particular case will exactly equal the observed value of the experiment (residual=0.) The sum of leverage values across all cases (design points) equals the number of coefficients (including the constant) fit by the model. The maximum leverage an experiment can have is 1/k, where k is the number of times the experiment is replicated. Values larger than 2 times the average leverage are flagged.

Internally Studentized Residual ($r_i = \dfrac{e_i}{s\sqrt{(1 - h_{ii})}}$):

The residual divided by the estimated standard deviation of that residual (dependent on leverage), which measures the number of standard deviations separating the actual from predicted values.

Externally Studentized Residual ($t_i = \dfrac{e_i}{s_{-1}\sqrt{1 - h_{ii}}}$):

This “outlier t” value is calculated by leaving the run in-question out of the analysis and estimating the response from the remaining runs. It represents the number of standard deviations between this predicted value and the actual response. Runs with large t values (rule-of-thumb: |t| > 3.5) are flagged and should be investigated.

Note: For verification runs, this is calculated using the predicted error (an expansion of the hat matrix).

DFFITS ($\text{DFFITS}_i = \dfrac{\hat{y}_i - \hat{y}_{i,-i}}{s_{-1}\sqrt{h_{ii}}}$, alternatively $\text{DFFITS}_i = t_i \left[\dfrac{h_{ii}}{1 - h_{ii}}\right]^{1/2}$) :

DFFFITS measures the influence each individual case (i) has on the predicted value (see Myers[1] page 284.) It is a function of leverage (h). Mathematically it is the studentized difference between the predicted value with and without observation "i". As shown by the alternative formula, DFFITS represents the externally studentized residual (t_i) magnified by high leverage points and shrunk by low leverage points. Note that DFFITs becomes undefined for leverages of one (h=1).

DFBETAS ($\mathrm{DFBETAS}_{j,i} = \frac{\beta_j - \hat{y}_{j,-i}}{s_{-1}\sqrt{c_{jj}}}$, c_{jj} is the j^{th} diagonal element of $(X'X)^{-1}$) :

DFBETAS measures the influence each individual case (i) has on each model coefficient (β_j). It represents the number of standard errors that the j^{th} beta-coefficient changes if the i^{th} observation is removed. Like DFFITS, this statistic becomes undefined for leverages of one (h=1). *DFBETAS are calculated for each beta-coefficient, so make sure to use the pull-down menu and click through the terms (the down arrow is a good shortcut key – also, try the wheel if you have one on your mouse).*

Cook's Distance ($D_i = \frac{1}{p} r_i^2 \left(\frac{h_{ii}}{1 - h_{ii}} \right)$):

A measure of how much the regression would change if the case is omitted from the analysis (see Weisberg[2] page 118). Relatively large values are associated with cases with high leverage and large studentized residuals. Cases with large D_i values relative to the other cases should be investigated. Look for mistakes in recording, an incorrect model, or a design point far from the others.

References:

1. Myers, Raymond: *Classical and Modern Regression with Applications, 2nd edition*, 2000, Duxbury Press.
2. Weisberg, Stanford: *Applied Linear Regression, 4th edition*, 2013, John Wiley & Sons, Inc.

Optimization Guide

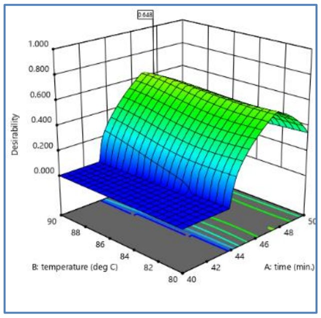
Numerical Optimization:

1. Analyze each response separately and establish an appropriate transformation and **model** for each. Be sure the fitted surface adequately represents your process. Check for:
    a) A significant model, i.e., a large F-value with $p<0.05$.
    b) Insignificant lack-of-fit, i.e., an F-value near one with $p>0.10$.
    c) Adequate precision, i.e., greater than 4.
    d) Well-behaved residuals.

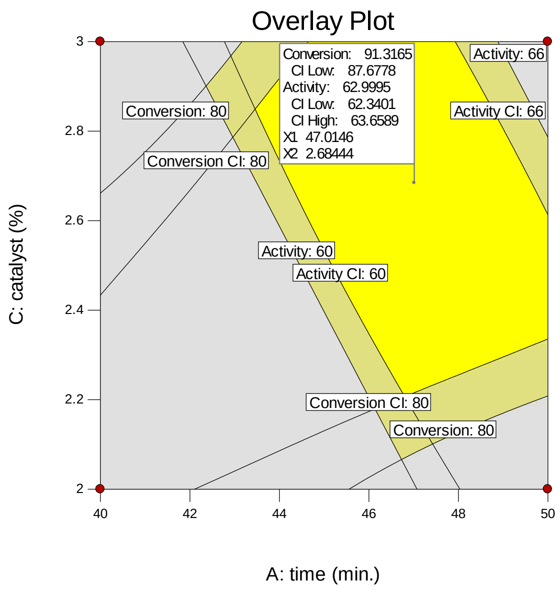
2. Set the following criteria for the **desirability** optimization:
    a) **Goal**: “maximize”, “minimize”, “target”, “in-range” and “C_pk”. Responses-only: “none” (default). Factors-only (default “in range”): “equal-to”.
    b) **Limits** lower and upper: Both ends required to establish the desirability from 0 or 1.
    c) **Weight** (optional): Enter 0.1 to 10 or drag the desirability ramp up (lighter) or down (heavier). The default of 1 keeps it linear. Weights >1 give more emphasis to the goal and vice-versa.
    d) **Importance** (optional): Changes goal’s importance less (+) to more (+++++) relative to the others (default +++).

3. Run the optimization (press **Solutions**).
    - **Report** shows settings of the factors, response values, and desirability for each solution from top to bottom.
    - **Ramps** show settings for all factors and the resulting predicted values for responses and where these fall on their desirability ramps. Cycle through rank of solution from top to bottom.
    - **Bar Graph** displays how well each variable satisfied their criterion.

4. **Graph** the desirability (shown) and the individual responses.



Graphical Optimization:



1. **Criteria** require at least one limit for at least one response:
2. Lower <u>only</u> if *maximized* (unlike numerical optimization where you must enter both lower and upper limits!)
    - ◆ Upper <u>only</u> if *minimized*
    - ◆ Lower <u>and</u> upper (specification range) if goal is *target*.
3. **Graph** the optimal point identified in the numerical optimization by clicking the #1 solution. It overlays all responses – shaded areas do <u>not</u> meet the specified criteria. The flagged window shows the "sweet spot". For a more conservative result, put in the confidence interval (CI) shown here or, for quality by design (QbD), the tolerance interval (TI).

<u>Suggestions for achieving desirable outcome:</u>

Numerical optimization provides powerful insights when combined with graphical analysis. However, it cannot substitute for subject matter knowledge. For example, you may define what you consider to be optimum, only to find zero desirability everywhere! To avoid finding no optimums, set broad optimization criteria and then narrow down as you gain knowledge. Most often, multiple passes are needed to find the "best" factor levels to simultaneously satisfy all operational constraints.

Inverses, 1st & 2nd Derivatives of Transformations

Transform	**Square root** *Counts*	**Log$_e$** *Variation*	**Log$_{10}$** *Variation*	**Inverse Square Root**
Power (lambda)	0.5	0	0	-0.5
Formula	$y' = \sqrt{y+k}$	$y' = \ln(y+k)$	$y' = \log(y+k)$	$y' = \frac{1}{\sqrt{y+k}}$
Inverse	$y = y'^2 - k$	$y = e^{y'} - k$	$y = 10^{y'} - k$	$y = y'^{-2} - k$
1st Derivative	$\frac{\partial y}{\partial y'} = 2y'$	$\frac{\partial y}{\partial y'} = \ln(e)e^{y'} = e^{y'}$	$\frac{\partial y}{\partial y'} = \ln(10)10^{y'}$	$\frac{\partial y}{\partial y'} = -2y'^{-3}$
2nd Derivative	$\frac{\partial^2 y}{\partial y'^2} = 2$	$\frac{\partial^2 y}{\partial y'^2} = e^{y'}$	$\frac{\partial^2 y}{\partial y'^2} = (\ln(10))^2 10^{y'}$	$\frac{\partial^2 y}{\partial y'^2} = 6y'^{-4}$

Transform	**Inverse** *Rates*	**Power** *when all else fails*	**ArcSin Square Root** *Binomial data* *y is a fraction (0-1)* y′ in radians	**Logit** *Asymptotically bounded data* LL=lower limit UL=upper limit
Power (lambda)	-1	λ	**NA**	**NA**
Formula	$y' = \frac{1}{y+k}$	$y' = (y+k)^{\lambda}$	$y' = \sin^{-1}\sqrt{y}$	$y' = \ln\left(\frac{y - LL}{UL - y}\right)$
Inverse	$y = y'^{-1} - k$	$y = (y')^{\frac{1}{\lambda}} + k$	$y = (\sin(y'))^2$	$y = \frac{UL(e^{y'}) + LL}{1 + e^{y'}}$
1st Derivative	$\frac{\partial y}{\partial y'} = -y'^{-2}$	$\frac{\partial y}{\partial y'} = \frac{1}{\lambda}(y')^{\left(\frac{1}{\lambda}-1\right)}$	$\frac{\partial y}{\partial y'} = 2\sin(y')\cos(y')$	$\frac{\partial y}{\partial y'} = \frac{e^{y'}(UL - LL)}{(1 + e^{y'})^2}$
2nd Derivative	$\frac{\partial^2 y}{\partial y'^2} = 2y'^{-3}$	$\frac{\partial^2 y}{\partial y'^2} = \frac{1}{\lambda}\left(\frac{1}{\lambda} - 1\right)(y')^{\left(\frac{1}{\lambda}-2\right)}$	$\frac{\partial^2 y}{\partial y'^2} = 2\cos(2y')$	$\frac{\partial^2 y}{\partial y'^2} = \frac{e^{y'}(1 - e^{y'})(UL - LL)}{(1 + e^{y'})^3}$

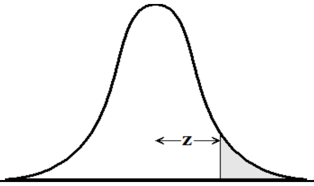
Section 3:

Appendix

Rev 1/15/24

Z-Table:

Tail area of unit normal distribution



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
3.6	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.7	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.8	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
3.9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

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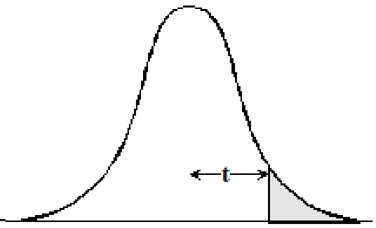
One-tailed / Two-tailed t-Table

Probability points of the t-distribution
with df degrees of freedom



	tail area probability									
1-tail	**0.40**	**0.25**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.0025**	**0.001**	**0.0005**
2-tail	**0.80**	**0.50**	**0.20**	**0.10**	**0.050**	**0.02**	**0.010**	**0.0050**	**0.002**	**0.0010**
df=1	0.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925	14.089	22.326	31.598
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.265	0.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.263	0.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.260	0.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.259	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.259	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.258	0.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.258	0.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.257	0.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.257	0.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.257	0.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.257	0.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.257	0.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.256	0.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.256	0.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	0.256	0.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.256	0.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.256	0.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.256	0.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.256	0.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.256	0.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	0.256	0.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.255	0.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	0.254	0.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	0.254	0.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.253	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

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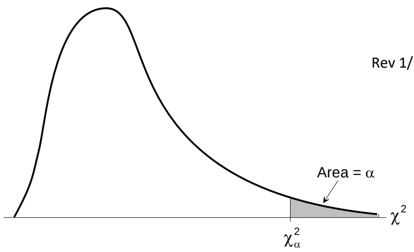
Cumulative Distribution of Chi-Square



Probability of a Greater value

df	**0.995**	**0.99**	**0.975**	**0.95**	**0.90**	**0.75**	**0.50**	**0.25**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**
1			0.001	0.004	0.016	0.102	0.455	1.32	2.71	3.84	5.02	6.64	7.88
2	0.010	0.020	0.051	0.103	0.211	0.575	1.39	2.77	4.61	5.99	7.38	9.21	10.60
3	0.072	0.115	0.216	0.352	0.584	1.21	2.37	4.11	6.25	7.82	9.35	11.35	12.84
4	0.207	0.297	0.484	0.711	1.06	1.92	3.36	5.39	7.78	9.49	11.14	13.28	14.86
5	0.412	0.554	0.831	1.15	1.61	2.68	4.35	6.63	9.24	11.07	12.83	15.09	16.75
6	0.676	0.872	1.24	1.64	2.20	3.46	5.35	7.84	10.65	12.59	14.45	16.81	18.55
7	0.989	1.24	1.69	2.17	2.83	4.26	6.35	9.04	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.22	13.36	15.51	17.54	20.09	21.96
9	1.74	2.09	2.70	3.33	4.17	5.90	8.34	11.39	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.55	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.58	5.58	7.58	10.34	13.70	17.28	19.68	21.92	24.73	26.76
12	3.07	3.57	4.40	5.23	6.30	8.44	11.34	14.85	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	9.30	12.34	15.98	19.81	22.36	24.74	27.69	29.82
14	4.08	4.66	5.63	6.57	7.79	10.17	13.34	17.12	21.06	23.69	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	11.04	14.34	18.25	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	11.91	15.34	19.37	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	12.79	16.34	20.49	24.77	27.59	30.19	33.41	35.72
18	6.27	7.02	8.23	9.39	10.87	13.68	17.34	21.61	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	14.56	18.34	22.72	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	15.45	19.34	23.83	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	16.34	20.34	24.94	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	17.24	21.34	26.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	18.14	22.34	27.14	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	19.04	23.34	28.24	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	19.94	24.34	29.34	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	20.84	25.34	30.44	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	21.75	26.34	31.53	36.74	40.11	43.20	46.96	49.65
28	12.46	13.57	15.31	16.93	18.94	22.66	27.34	32.62	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	23.57	28.34	33.71	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	24.48	29.34	34.80	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	33.66	39.34	45.62	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	42.94	49.34	56.33	63.17	67.51	71.42	76.15	79.49
60	35.53	37.49	40.48	43.19	46.46	52.29	59.34	66.98	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	61.70	69.33	77.58	85.53	90.53	95.02	100.43	104.22
80	51.17	53.54	57.15	60.39	64.28	71.15	79.33	88.13	96.58	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	73.29	80.63	89.33	98.65	107.57	113.15	118.14	124.12	128.30
100	67.33	70.07	74.22	77.93	82.36	90.13	99.33	109.14	118.50	124.34	129.56	135.81	140.17

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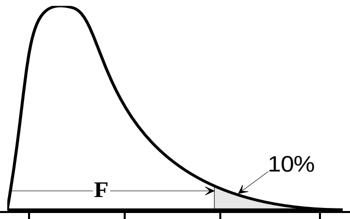
F-Table for 10%



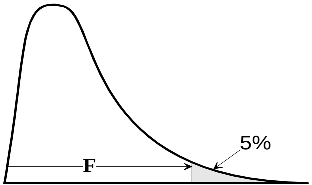
Percentage points of the F-distribution: upper 10% points

df_{num} df_{den}	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**15**	**20**
1	39.863	49.500	53.593	55.833	57.240	58.204	58.906	59.439	59.858	60.195	61.220	61.740
2	8.526	9.000	9.162	9.243	9.293	9.326	9.349	9.367	9.381	9.392	9.425	9.441
3	5.538	5.462	5.391	5.343	5.309	5.285	5.266	5.252	5.240	5.230	5.200	5.184
4	4.545	4.325	4.191	4.107	4.051	4.010	3.979	3.955	3.936	3.920	3.870	3.844
5	4.060	3.780	3.619	3.520	3.453	3.405	3.368	3.339	3.316	3.297	3.238	3.207
6	3.776	3.463	3.289	3.181	3.108	3.055	3.014	2.983	2.958	2.937	2.871	2.836
7	3.589	3.257	3.074	2.961	2.883	2.827	2.785	2.752	2.725	2.703	2.632	2.595
8	3.458	3.113	2.924	2.806	2.726	2.668	2.624	2.589	2.561	2.538	2.464	2.425
9	3.360	3.006	2.813	2.693	2.611	2.551	2.505	2.469	2.440	2.416	2.340	2.298
10	3.285	2.924	2.728	2.605	2.522	2.461	2.414	2.377	2.347	2.323	2.244	2.201
11	3.225	2.860	2.660	2.536	2.451	2.389	2.342	2.304	2.274	2.248	2.167	2.123
12	3.177	2.807	2.606	2.480	2.394	2.331	2.283	2.245	2.214	2.188	2.105	2.060
13	3.136	2.763	2.560	2.434	2.347	2.283	2.234	2.195	2.164	2.138	2.053	2.007
14	3.102	2.726	2.522	2.395	2.307	2.243	2.193	2.154	2.122	2.095	2.010	1.962
15	3.073	2.695	2.490	2.361	2.273	2.208	2.158	2.119	2.086	2.059	1.972	1.924
16	3.048	2.668	2.462	2.333	2.244	2.178	2.128	2.088	2.055	2.028	1.940	1.891
17	3.026	2.645	2.437	2.308	2.218	2.152	2.102	2.061	2.028	2.001	1.912	1.862
18	3.007	2.624	2.416	2.286	2.196	2.130	2.079	2.038	2.005	1.977	1.887	1.837
19	2.990	2.606	2.397	2.266	2.176	2.109	2.058	2.017	1.984	1.956	1.865	1.814
20	2.975	2.589	2.380	2.249	2.158	2.091	2.040	1.999	1.965	1.937	1.845	1.794
21	2.961	2.575	2.365	2.233	2.142	2.075	2.023	1.982	1.948	1.920	1.827	1.776
22	2.949	2.561	2.351	2.219	2.128	2.060	2.008	1.967	1.933	1.904	1.811	1.759
23	2.937	2.549	2.339	2.207	2.115	2.047	1.995	1.953	1.919	1.890	1.796	1.744
24	2.927	2.538	2.327	2.195	2.103	2.035	1.983	1.941	1.906	1.877	1.783	1.730
25	2.918	2.528	2.317	2.184	2.092	2.024	1.971	1.929	1.895	1.866	1.771	1.718
26	2.909	2.519	2.307	2.174	2.082	2.014	1.961	1.919	1.884	1.855	1.760	1.706
27	2.901	2.511	2.299	2.165	2.073	2.005	1.952	1.909	1.874	1.845	1.749	1.695
28	2.894	2.503	2.291	2.157	2.064	1.996	1.943	1.900	1.865	1.836	1.740	1.685
29	2.887	2.495	2.283	2.149	2.057	1.988	1.935	1.892	1.857	1.827	1.731	1.676
30	2.881	2.489	2.276	2.142	2.049	1.980	1.927	1.884	1.849	1.819	1.722	1.667
40	2.835	2.440	2.226	2.091	1.997	1.927	1.873	1.829	1.793	1.763	1.662	1.605
60	2.791	2.393	2.177	2.041	1.946	1.875	1.819	1.775	1.738	1.707	1.603	1.543
120	2.748	2.347	2.130	1.992	1.896	1.824	1.767	1.722	1.684	1.652	1.545	1.482
100K	2.706	2.303	2.084	1.945	1.847	1.774	1.717	1.670	1.632	1.599	1.487	1.421

K (Multiply this value by 1000)

F-Table for 5%

Percentage points of the F-distribution:
upper 5% points



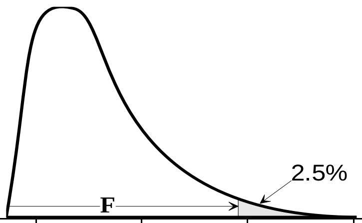
df_{num} df_{den}	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**15**	**20**
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	245.95	248.01
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385	19.396	19.429	19.446
3	10.128	9.552	9.277	9.117	9.013	8.941	8.887	8.845	8.812	8.786	8.703	8.660
4	7.709	6.944	6.591	6.388	6.256	6.163	6.094	6.041	5.999	5.964	5.858	5.803
5	6.608	5.786	5.409	5.192	5.050	4.950	4.876	4.818	4.772	4.735	4.619	4.558
6	5.987	5.143	4.757	4.534	4.387	4.284	4.207	4.147	4.099	4.060	3.938	3.874
7	5.591	4.737	4.347	4.120	3.972	3.866	3.787	3.726	3.677	3.637	3.511	3.445
8	5.318	4.459	4.066	3.838	3.687	3.581	3.500	3.438	3.388	3.347	3.218	3.150
9	5.117	4.256	3.863	3.633	3.482	3.374	3.293	3.230	3.179	3.137	3.006	2.936
10	4.965	4.103	3.708	3.478	3.326	3.217	3.135	3.072	3.020	2.978	2.845	2.774
11	4.844	3.982	3.587	3.357	3.204	3.095	3.012	2.948	2.896	2.854	2.719	2.646
12	4.747	3.885	3.490	3.259	3.106	2.996	2.913	2.849	2.796	2.753	2.617	2.544
13	4.667	3.806	3.411	3.179	3.025	2.915	2.832	2.767	2.714	2.671	2.533	2.459
14	4.600	3.739	3.344	3.112	2.958	2.848	2.764	2.699	2.646	2.602	2.463	2.388
15	4.543	3.682	3.287	3.056	2.901	2.790	2.707	2.641	2.588	2.544	2.403	2.328
16	4.494	3.634	3.239	3.007	2.852	2.741	2.657	2.591	2.538	2.494	2.352	2.276
17	4.451	3.592	3.197	2.965	2.810	2.699	2.614	2.548	2.494	2.450	2.308	2.230
18	4.414	3.555	3.160	2.928	2.773	2.661	2.577	2.510	2.456	2.412	2.269	2.191
19	4.381	3.522	3.127	2.895	2.740	2.628	2.544	2.477	2.423	2.378	2.234	2.155
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348	2.203	2.124
21	4.325	3.467	3.072	2.840	2.685	2.573	2.488	2.420	2.366	2.321	2.176	2.096
22	4.301	3.443	3.049	2.817	2.661	2.549	2.464	2.397	2.342	2.297	2.151	2.071
23	4.279	3.422	3.028	2.796	2.640	2.528	2.442	2.375	2.320	2.275	2.128	2.048
24	4.260	3.403	3.009	2.776	2.621	2.508	2.423	2.355	2.300	2.255	2.108	2.027
25	4.242	3.385	2.991	2.759	2.603	2.490	2.405	2.337	2.282	2.236	2.089	2.007
26	4.225	3.369	2.975	2.743	2.587	2.474	2.388	2.321	2.265	2.220	2.072	1.990
27	4.210	3.354	2.960	2.728	2.572	2.459	2.373	2.305	2.250	2.204	2.056	1.974
28	4.196	3.340	2.947	2.714	2.558	2.445	2.359	2.291	2.236	2.190	2.041	1.959
29	4.183	3.328	2.934	2.701	2.545	2.432	2.346	2.278	2.223	2.177	2.027	1.945
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334	2.266	2.211	2.165	2.015	1.932
40	4.085	3.232	2.839	2.606	2.449	2.336	2.249	2.180	2.124	2.077	1.924	1.839
60	4.001	3.150	2.758	2.525	2.368	2.254	2.167	2.097	2.040	1.993	1.836	1.748
120	3.920	3.072	2.680	2.447	2.290	2.175	2.087	2.016	1.959	1.910	1.750	1.659
100K	3.842	2.996	2.605	2.372	2.214	2.099	2.010	1.939	1.880	1.831	1.666	1.571

K (Multiply this value by 1000)

Rev 1/15/24

F-Table for 2.5%

Percentage points of the F-distribution: upper 2.5% points



df_{num} df_{den}	1	2	3	4	5	6	7	8	9	10	15	20
1	647.79	799.50	864.16	899.58	921.85	937.11	948.22	956.66	963.28	968.63	984.87	993.10
2	38.506	39.000	39.165	39.248	39.298	39.331	39.355	39.373	39.387	39.398	39.431	39.448
3	17.443	16.044	15.439	15.101	14.885	14.735	14.624	14.540	14.473	14.419	14.253	14.167
4	12.218	10.649	9.979	9.605	9.364	9.197	9.074	8.980	8.905	8.844	8.657	8.560
5	10.007	8.434	7.764	7.388	7.146	6.978	6.853	6.757	6.681	6.619	6.428	6.329
6	8.813	7.260	6.599	6.227	5.988	5.820	5.695	5.600	5.523	5.461	5.269	5.168
7	8.073	6.542	5.890	5.523	5.285	5.119	4.995	4.899	4.823	4.761	4.568	4.467
8	7.571	6.059	5.416	5.053	4.817	4.652	4.529	4.433	4.357	4.295	4.101	3.999
9	7.209	5.715	5.078	4.718	4.484	4.320	4.197	4.102	4.026	3.964	3.769	3.667
10	6.937	5.456	4.826	4.468	4.236	4.072	3.950	3.855	3.779	3.717	3.522	3.419
11	6.724	5.256	4.630	4.275	4.044	3.881	3.759	3.664	3.588	3.526	3.330	3.226
12	6.554	5.096	4.474	4.121	3.891	3.728	3.607	3.512	3.436	3.374	3.177	3.073
13	6.414	4.965	4.347	3.996	3.767	3.604	3.483	3.388	3.312	3.250	3.053	2.948
14	6.298	4.857	4.242	3.892	3.663	3.501	3.380	3.285	3.209	3.147	2.949	2.844
15	6.200	4.765	4.153	3.804	3.576	3.415	3.293	3.199	3.123	3.060	2.862	2.756
16	6.115	4.687	4.077	3.729	3.502	3.341	3.219	3.125	3.049	2.986	2.788	2.681
17	6.042	4.619	4.011	3.665	3.438	3.277	3.156	3.061	2.985	2.922	2.723	2.616
18	5.978	4.560	3.954	3.608	3.382	3.221	3.100	3.005	2.929	2.866	2.667	2.559
19	5.922	4.508	3.903	3.559	3.333	3.172	3.051	2.956	2.880	2.817	2.617	2.509
20	5.871	4.461	3.859	3.515	3.289	3.128	3.007	2.913	2.837	2.774	2.573	2.464
21	5.827	4.420	3.819	3.475	3.250	3.090	2.969	2.874	2.798	2.735	2.534	2.425
22	5.786	4.383	3.783	3.440	3.215	3.055	2.934	2.839	2.763	2.700	2.498	2.389
23	5.750	4.349	3.750	3.408	3.183	3.023	2.902	2.808	2.731	2.668	2.466	2.357
24	5.717	4.319	3.721	3.379	3.155	2.995	2.874	2.779	2.703	2.640	2.437	2.327
25	5.686	4.291	3.694	3.353	3.129	2.969	2.848	2.753	2.677	2.613	2.411	2.300
26	5.659	4.265	3.670	3.329	3.105	2.945	2.824	2.729	2.653	2.590	2.387	2.276
27	5.633	4.242	3.647	3.307	3.083	2.923	2.802	2.707	2.631	2.568	2.364	2.253
28	5.610	4.221	3.626	3.286	3.063	2.903	2.782	2.687	2.611	2.547	2.344	2.232
29	5.588	4.201	3.607	3.267	3.044	2.884	2.763	2.669	2.592	2.529	2.325	2.213
30	5.568	4.182	3.589	3.250	3.026	2.867	2.746	2.651	2.575	2.511	2.307	2.195
40	5.424	4.051	3.463	3.126	2.904	2.744	2.624	2.529	2.452	2.388	2.182	2.068
60	5.286	3.925	3.343	3.008	2.786	2.627	2.507	2.412	2.334	2.270	2.061	1.944
120	5.152	3.805	3.227	2.894	2.674	2.515	2.395	2.299	2.222	2.157	1.945	1.825
∞	5.024	3.689	3.116	2.786	2.567	2.408	2.288	2.192	2.114	2.048	1.833	1.709

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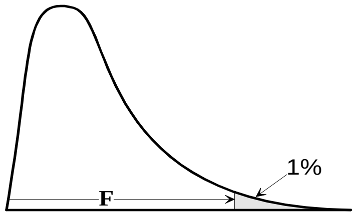
F-Table for 1%



Percentage points of the F-distribution:
upper 1% points

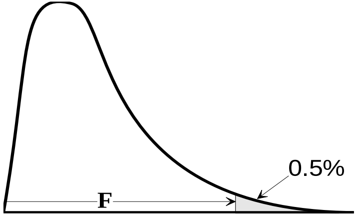
df_{num} df_{den}	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**15**	**20**
1	4052.2	4999.5	5403.3	5624.6	5763.6	5859.0	5928.3	5981.1	6022.5	6055.8	6157.3	6208.7
2	98.503	99.000	99.166	99.249	99.299	99.333	99.356	99.374	99.388	99.399	99.433	99.449
3	34.116	30.817	29.457	28.710	28.237	27.911	27.672	27.489	27.345	27.229	26.872	26.690
4	21.198	18.000	16.694	15.977	15.522	15.207	14.976	14.799	14.659	14.546	14.198	14.020
5	16.258	13.274	12.060	11.392	10.967	10.672	10.456	10.289	10.158	10.051	9.722	9.553
6	13.745	10.925	9.780	9.148	8.746	8.466	8.260	8.102	7.976	7.874	7.559	7.396
7	12.246	9.547	8.451	7.847	7.460	7.191	6.993	6.840	6.719	6.620	6.314	6.155
8	11.259	8.649	7.591	7.006	6.632	6.371	6.178	6.029	5.911	5.814	5.515	5.359
9	10.561	8.022	6.992	6.422	6.057	5.802	5.613	5.467	5.351	5.257	4.962	4.808
10	10.044	7.559	6.552	5.994	5.636	5.386	5.200	5.057	4.942	4.849	4.558	4.405
11	9.646	7.206	6.217	5.668	5.316	5.069	4.886	4.744	4.632	4.539	4.251	4.099
12	9.330	6.927	5.953	5.412	5.064	4.821	4.640	4.499	4.388	4.296	4.010	3.858
13	9.074	6.701	5.739	5.205	4.862	4.620	4.441	4.302	4.191	4.100	3.815	3.665
14	8.862	6.515	5.564	5.035	4.695	4.456	4.278	4.140	4.030	3.939	3.656	3.505
15	8.683	6.359	5.417	4.893	4.556	4.318	4.142	4.004	3.895	3.805	3.522	3.372
16	8.531	6.226	5.292	4.773	4.437	4.202	4.026	3.890	3.780	3.691	3.409	3.259
17	8.400	6.112	5.185	4.669	4.336	4.102	3.927	3.791	3.682	3.593	3.312	3.162
18	8.285	6.013	5.092	4.579	4.248	4.015	3.841	3.705	3.597	3.508	3.227	3.077
19	8.185	5.926	5.010	4.500	4.171	3.939	3.765	3.631	3.523	3.434	3.153	3.003
20	8.096	5.849	4.938	4.431	4.103	3.871	3.699	3.564	3.457	3.368	3.088	2.938
21	8.017	5.780	4.874	4.369	4.042	3.812	3.640	3.506	3.398	3.310	3.030	2.880
22	7.945	5.719	4.817	4.313	3.988	3.758	3.587	3.453	3.346	3.258	2.978	2.827
23	7.881	5.664	4.765	4.264	3.939	3.710	3.539	3.406	3.299	3.211	2.931	2.781
24	7.823	5.614	4.718	4.218	3.895	3.667	3.496	3.363	3.256	3.168	2.889	2.738
25	7.770	5.568	4.675	4.177	3.855	3.627	3.457	3.324	3.217	3.129	2.850	2.699
26	7.721	5.526	4.637	4.140	3.818	3.591	3.421	3.288	3.182	3.094	2.815	2.664
27	7.677	5.488	4.601	4.106	3.785	3.558	3.388	3.256	3.149	3.062	2.783	2.632
28	7.636	5.453	4.568	4.074	3.754	3.528	3.358	3.226	3.120	3.032	2.753	2.602
29	7.598	5.420	4.538	4.045	3.725	3.499	3.330	3.198	3.092	3.005	2.726	2.574
30	7.562	5.390	4.510	4.018	3.699	3.473	3.304	3.173	3.067	2.979	2.700	2.549
40	7.314	5.179	4.313	3.828	3.514	3.291	3.124	2.993	2.888	2.801	2.522	2.369
60	7.077	4.977	4.126	3.649	3.339	3.119	2.953	2.823	2.718	2.632	2.352	2.198
120	6.851	4.787	3.949	3.480	3.174	2.956	2.792	2.663	2.559	2.472	2.192	2.035
100K	6.635	4.605	3.782	3.319	3.017	2.802	2.640	2.511	2.408	2.321	2.039	1.878

K (Multiply this value by 1000)

Rev 1/15/24

F-Table for 0.5%

Percentage points of the F-distribution:
upper 0.5% points

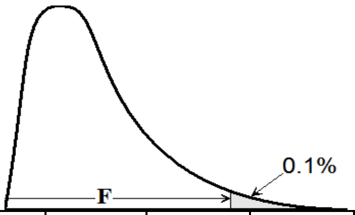


df_{num} df_{den}	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**15**	**20**
1	16210.7	19999.5	21614.7	22499.6	23055.8	23437.1	23714.6	23925.6	24091.0	24224.5	24630.2	24836.0
2	198.50	199.00	199.17	199.25	199.30	199.33	199.36	199.37	199.39	199.40	199.43	199.45
3	55.552	49.799	47.467	46.195	45.392	44.838	44.434	44.126	43.882	43.686	43.085	42.778
4	31.333	26.284	24.259	23.155	22.456	21.975	21.622	21.352	21.139	20.967	20.438	20.167
5	22.785	18.314	16.530	15.556	14.940	14.513	14.200	13.961	13.772	13.618	13.146	12.903
6	18.635	14.544	12.917	12.028	11.464	11.073	10.786	10.566	10.391	10.250	9.814	9.589
7	16.236	12.404	10.882	10.050	9.522	9.155	8.885	8.678	8.514	8.380	7.968	7.754
8	14.688	11.042	9.596	8.805	8.302	7.952	7.694	7.496	7.339	7.211	6.814	6.608
9	13.614	10.107	8.717	7.956	7.471	7.134	6.885	6.693	6.541	6.417	6.032	5.832
10	12.826	9.427	8.081	7.343	6.872	6.545	6.302	6.116	5.968	5.847	5.471	5.274
11	12.226	8.912	7.600	6.881	6.422	6.102	5.865	5.682	5.537	5.418	5.049	4.855
12	11.754	8.510	7.226	6.521	6.071	5.757	5.525	5.345	5.202	5.085	4.721	4.530
13	11.374	8.186	6.926	6.233	5.791	5.482	5.253	5.076	4.935	4.820	4.460	4.270
14	11.060	7.922	6.680	5.998	5.562	5.257	5.031	4.857	4.717	4.603	4.247	4.059
15	10.798	7.701	6.476	5.803	5.372	5.071	4.847	4.674	4.536	4.424	4.070	3.883
16	10.575	7.514	6.303	5.638	5.212	4.913	4.692	4.521	4.384	4.272	3.920	3.734
17	10.384	7.354	6.156	5.497	5.075	4.779	4.559	4.389	4.254	4.142	3.793	3.607
18	10.218	7.215	6.028	5.375	4.956	4.663	4.445	4.276	4.141	4.030	3.683	3.498
19	10.073	7.093	5.916	5.268	4.853	4.561	4.345	4.177	4.043	3.933	3.587	3.402
20	9.944	6.986	5.818	5.174	4.762	4.472	4.257	4.090	3.956	3.847	3.502	3.318
21	9.830	6.891	5.730	5.091	4.681	4.393	4.179	4.013	3.880	3.771	3.427	3.243
22	9.727	6.806	5.652	5.017	4.609	4.322	4.109	3.944	3.812	3.703	3.360	3.176
23	9.635	6.730	5.582	4.950	4.544	4.259	4.047	3.882	3.750	3.642	3.300	3.116
24	9.551	6.661	5.519	4.890	4.486	4.202	3.991	3.826	3.695	3.587	3.246	3.062
25	9.475	6.598	5.462	4.835	4.433	4.150	3.939	3.776	3.645	3.537	3.196	3.013
26	9.406	6.541	5.409	4.785	4.384	4.103	3.893	3.730	3.599	3.492	3.151	2.968
27	9.342	6.489	5.361	4.740	4.340	4.059	3.850	3.687	3.557	3.450	3.110	2.928
28	9.284	6.440	5.317	4.698	4.300	4.020	3.811	3.649	3.519	3.412	3.073	2.890
29	9.230	6.396	5.276	4.659	4.262	3.983	3.775	3.613	3.483	3.377	3.038	2.855
30	9.180	6.355	5.239	4.623	4.228	3.949	3.742	3.580	3.450	3.344	3.006	2.823
40	8.828	6.066	4.976	4.374	3.986	3.713	3.509	3.350	3.222	3.117	2.781	2.598
60	8.495	5.795	4.729	4.140	3.760	3.492	3.291	3.134	3.008	2.904	2.570	2.387
120	8.179	5.539	4.497	3.921	3.548	3.285	3.087	2.933	2.808	2.705	2.373	2.188
∞	7.880	5.299	4.280	3.715	3.350	3.091	2.897	2.745	2.621	2.519	2.187	2.000

Rev 1/15/24

F-Table for 0.1%

Percentage points of the F-distribution:
upper 0.1% points



df_{num} df_{den}	1	2	3	4	5	6	7	8	9	10	15	20
1	405.2K	500.0K	540.4K	562.5K3	576.4K	585.9K	592.9K	598.1K	602.3K9	605.6K	615.8K	620.9K
2	998.50	999.00	999.17	999.25	999.30	999.33	999.36	999.37	999.39	999.40	999.43	999.45
3	167.03	148.50	141.11	137.10	134.58	132.85	131.58	130.62	129.86	129.25	127.37	126.42
4	74.137	61.246	56.177	53.436	51.712	50.525	49.658	48.996	48.475	48.053	46.761	46.100
5	47.181	37.122	33.202	31.085	29.752	28.834	28.163	27.649	27.244	26.917	25.911	25.395
6	35.507	27.000	23.703	21.924	20.803	20.030	19.463	19.030	18.688	18.411	17.559	17.120
7	29.245	21.689	18.772	17.198	16.206	15.521	15.019	14.634	14.330	14.083	13.324	12.932
8	25.415	18.494	15.829	14.392	13.485	12.858	12.398	12.046	11.767	11.540	10.841	10.480
9	22.857	16.387	13.902	12.560	11.714	11.128	10.698	10.368	10.107	9.894	9.238	8.898
10	21.040	14.905	12.553	11.283	10.481	9.926	9.517	9.204	8.956	8.754	8.129	7.804
11	19.687	13.812	11.561	10.346	9.578	9.047	8.655	8.355	8.116	7.922	7.321	7.008
12	18.643	12.974	10.804	9.633	8.892	8.379	8.001	7.710	7.480	7.292	6.709	6.405
13	17.815	12.313	10.209	9.073	8.354	7.856	7.489	7.206	6.982	6.799	6.231	5.934
14	17.143	11.779	9.729	8.622	7.922	7.436	7.077	6.802	6.583	6.404	5.848	5.557
15	16.587	11.339	9.335	8.253	7.567	7.092	6.741	6.471	6.256	6.081	5.535	5.248
16	16.120	10.971	9.006	7.944	7.272	6.805	6.460	6.195	5.984	5.812	5.274	4.992
17	15.722	10.658	8.727	7.683	7.022	6.562	6.223	5.962	5.754	5.584	5.054	4.775
18	15.379	10.390	8.487	7.459	6.808	6.355	6.021	5.763	5.558	5.390	4.866	4.590
19	15.081	10.157	8.280	7.265	6.622	6.175	5.845	5.590	5.388	5.222	4.704	4.430
20	14.819	9.953	8.098	7.096	6.461	6.019	5.692	5.440	5.239	5.075	4.562	4.290
21	14.587	9.772	7.938	6.947	6.318	5.881	5.557	5.308	5.109	4.946	4.437	4.167
22	14.380	9.612	7.796	6.814	6.191	5.758	5.438	5.190	4.993	4.832	4.326	4.058
23	14.195	9.469	7.669	6.696	6.078	5.649	5.331	5.085	4.890	4.730	4.227	3.961
24	14.028	9.339	7.554	6.589	5.977	5.550	5.235	4.991	4.797	4.638	4.139	3.873
25	13.877	9.223	7.451	6.493	5.885	5.462	5.148	4.906	4.713	4.555	4.059	3.794
26	13.739	9.116	7.357	6.406	5.802	5.381	5.070	4.829	4.637	4.480	3.986	3.723
27	13.613	9.019	7.272	6.326	5.726	5.308	4.998	4.759	4.568	4.412	3.920	3.658
28	13.498	8.931	7.193	6.253	5.656	5.241	4.933	4.695	4.505	4.349	3.859	3.598
29	13.391	8.849	7.121	6.186	5.593	5.179	4.873	4.636	4.447	4.292	3.804	3.543
30	13.293	8.773	7.054	6.125	5.534	5.122	4.817	4.581	4.393	4.239	3.753	3.493
40	12.609	8.251	6.595	5.698	5.128	4.731	4.436	4.207	4.024	3.874	3.400	3.145
60	11.973	7.768	6.171	5.307	4.757	4.372	4.086	3.865	3.687	3.541	3.078	2.827
120	11.380	7.321	5.781	4.947	4.416	4.044	3.767	3.552	3.379	3.237	2.783	2.534
100K	10.828	6.908	5.422	4.617	4.103	3.743	3.475	3.266	3.098	2.959	2.513	2.266

K (Multiply entries by 1000 in first row of F values, and last value for df)

Factors for Two-Sided Tolerance Limits for Normal Distributions

Factors K_2 such that the probability is γ (gamma) that at least a proportion P
of the distribution will be included between $\bar{Y} \pm K_2 s$.

$\gamma = 0.75$					
n	**P = 0.75**	**P = 0.90**	**P = 0.95**	**P = 0.99**	**P = 0.999**
2	4.498	6.301	7.414	9.531	11.920
3	2.501	3.538	4.187	5.431	6.844
4	2.035	2.892	3.431	4.471	5.657
5	1.825	2.599	3.088	4.033	5.117
6	1.704	2.429	2.889	3.779	4.802
7	1.624	2.318	2.757	3.611	4.593
8	1.568	2.238	2.663	3.491	4.444
9	1.525	2.178	2.593	3.400	4.330
10	1.492	2.131	2.537	3.328	4.241
11	1.465	2.093	2.493	3.271	4.169
12	1.443	2.062	2.456	3.223	4.110
13	1.425	2.036	2.424	3.183	4.059
14	1.409	2.013	2.398	3.148	4.016
15	1.395	1.994	2.375	3.118	3.979
16	1.383	1.977	2.355	3.092	3.946
17	1.372	1.962	2.337	3.069	3.917
18	1.363	1.948	2.321	3.048	3.891
19	1.355	1.936	2.307	3.030	3.867
20	1.347	1.925	2.294	3.013	3.846
21	1.340	1.915	2.282	2.998	3.827
22	1.334	1.906	2.271	2.984	3.809
23	1.328	1.898	2.261	2.971	3.793
24	1.322	1.891	2.252	2.959	3.778
25	1.317	1.883	2.244	2.948	3.764
26	1.313	1.877	2.236	2.938	3.751
27	1.309	1.871	2.229	2.929	3.739
28	1.305	1.865	2.222	2.920	3.728
29	1.301	1.860	2.216	2.911	3.718
30	1.297	1.855	2.210	2.904	3.708
35	1.283	1.834	2.185	2.871	3.667
40	1.271	1.818	2.166	2.846	3.635
45	1.262	1.805	2.150	2.826	3.609
50	1.255	1.794	2.138	2.809	3.588
75	1.231	1.760	2.098	2.756	3.521
100	1.218	1.742	2.075	2.727	3.484
200	1.195	1.709	2.037	2.677	3.419
500	1.177	1.683	2.006	2.636	3.368
1000	1.169	1.671	1.992	2.617	3.344
∞	1.150	1.645	1.960	2.576	3.291

Factors for Two-Sided Tolerance Limits for Normal Distributions

Factors K_2 such that the probability is γ (gamma) that at least a proportion P
of the distribution will be included between $\bar{Y} \pm K_2 s$.

$\gamma = 0.90$					
n	**P = 0.75**	**P = 0.90**	**P = 0.95**	**P = 0.99**	**P = 0.999**
2	11.407	15.978	18.800	24.167	30.227
3	4.132	5.847	6.919	8.974	11.309
4	2.932	4.166	4.943	6.440	8.149
5	2.454	3.494	4.152	5.423	6.879
6	2.196	3.131	3.723	4.870	6.188
7	2.034	2.902	3.452	4.521	5.750
8	1.921	2.743	3.264	4.278	5.446
9	1.839	2.626	3.125	4.098	5.220
10	1.775	2.535	3.018	3.959	5.046
11	1.724	2.463	2.933	3.849	4.906
12	1.683	2.404	2.863	3.758	4.792
13	1.648	2.355	2.805	3.682	4.697
14	1.619	2.314	2.756	3.618	4.615
15	1.594	2.278	2.713	3.562	4.545
16	1.572	2.246	2.676	3.514	4.484
17	1.552	2.219	2.643	3.471	4.430
18	1.535	2.194	2.614	3.433	4.382
19	1.520	2.172	2.588	3.399	4.339
20	1.506	2.152	2.564	3.368	4.300
21	1.493	2.135	2.543	3.340	4.264
22	1.482	2.118	2.524	3.315	4.232
23	1.471	2.103	2.506	3.292	4.203
24	1.462	2.089	2.489	3.270	4.176
25	1.453	2.077	2.474	3.251	4.151
26	1.444	2.065	2.460	3.232	4.127
27	1.437	2.054	2.447	3.215	4.106
28	1.430	2.044	2.435	3.199	4.085
29	1.423	2.034	2.424	3.184	4.066
30	1.417	2.025	2.413	3.170	4.049
35	1.390	1.988	2.368	3.112	3.974
40	1.370	1.959	2.334	3.066	3.917
45	1.354	1.935	2.306	3.030	3.871
50	1.340	1.916	2.284	3.001	3.833
75	1.298	1.856	2.211	2.906	3.712
100	1.275	1.822	2.172	2.854	3.646
200	1.234	1.764	2.102	2.762	3.529
500	1.201	1.717	2.046	2.689	3.434
1000	1.185	1.695	2.019	2.654	3.390
∞	1.150	1.645	1.960	2.576	3.291

Factors for Two-Sided Tolerance Limits for Normal Distributions

Factors K_2 such that the probability is γ (gamma) that at least a proportion P
of the distribution will be included between $\bar{Y} \pm K_2 s$.

$\gamma = 0.95$					
n	**P = 0.75**	**P = 0.90**	**P = 0.95**	**P = 0.99**	**P = 0.999**
2	22.858	32.019	37.674	48.430	60.573
3	5.922	8.380	9.916	12.861	16.208
4	3.779	5.369	6.370	8.299	10.502
5	3.002	4.275	5.079	6.634	8.415
6	2.604	3.712	4.414	5.775	7.337
7	2.361	3.369	4.007	5.248	6.676
8	2.197	3.136	3.732	4.891	6.226
9	2.078	2.967	3.532	4.631	5.899
10	1.987	2.839	3.379	4.433	5.649
11	1.916	2.737	3.259	4.277	5.452
12	1.858	2.655	3.162	4.150	5.291
13	1.810	2.587	3.081	4.044	5.158
14	1.770	2.529	3.012	3.955	5.045
15	1.735	2.480	2.954	3.878	4.949
16	1.705	2.437	2.903	3.812	4.865
17	1.679	2.400	2.858	3.754	4.791
18	1.655	2.366	2.819	3.702	4.725
19	1.635	2.337	2.784	3.656	4.667
20	1.616	2.310	2.752	3.615	4.614
21	1.599	2.286	2.723	3.577	4.567
22	1.584	2.264	2.697	3.543	4.523
23	1.570	2.244	2.673	3.512	4.484
24	1.557	2.225	2.651	3.483	4.447
25	1.545	2.208	2.631	3.457	4.413
26	1.534	2.193	2.612	3.432	4.382
27	1.523	2.178	2.595	3.409	4.353
28	1.514	2.164	2.579	3.388	4.326
29	1.505	2.152	2.564	3.368	4.301
30	1.497	2.140	2.549	3.350	4.278
35	1.462	2.090	2.490	3.272	4.179
40	1.435	2.052	2.445	3.212	4.103
45	1.414	2.021	2.408	3.165	4.042
50	1.396	1.996	2.379	3.126	3.993
75	1.341	1.917	2.285	3.002	3.835
100	1.311	1.874	2.233	2.934	3.748
200	1.258	1.798	2.143	2.816	3.597
500	1.215	1.737	2.070	2.721	3.475
1000	1.195	1.709	2.036	2.676	3.418
∞	1.150	1.645	1.960	2.576	3.291

Factors for Two-Sided Tolerance Limits for Normal Distributions

Factors K_2 such that the probability is γ (gamma) that at least a proportion P of the distribution will be included between $\bar{Y} \pm K_2 s$.

$\gamma = 0.99$					
n	**P = 0.75**	**P = 0.90**	**P = 0.95**	**P = 0.99**	**P = 0.999**
2	114.363	160.193	188.491	242.300	303.054
3	13.378	18.930	22.401	29.055	36.616
4	6.614	9.398	11.150	14.527	18.383
5	4.643	6.612	7.855	10.260	13.015
6	3.743	5.337	6.345	8.301	10.548
7	3.233	4.613	5.488	7.187	9.142
8	2.905	4.147	4.936	6.468	8.234
9	2.677	3.822	4.550	5.966	7.600
10	2.508	3.582	4.265	5.594	7.129
11	2.378	3.397	4.045	5.308	6.766
12	2.274	3.250	3.870	5.079	6.477
13	2.190	3.130	3.727	4.893	6.240
14	2.120	3.029	3.608	4.737	6.043
15	2.060	2.945	3.507	4.605	5.876
16	2.009	2.872	3.421	4.492	5.732
17	1.965	2.808	3.345	4.393	5.607
18	1.926	2.753	3.279	4.307	5.497
19	1.891	2.703	3.221	4.230	5.399
20	1.860	2.659	3.168	4.161	5.312
21	1.833	2.620	3.121	4.100	5.234
22	1.808	2.584	3.078	4.044	5.163
23	1.785	2.551	3.040	3.993	5.098
24	1.764	2.522	3.004	3.947	5.039
25	1.745	2.494	2.972	3.904	4.985
26	1.727	2.469	2.941	3.865	4.935
27	1.711	2.446	2.914	3.828	4.888
28	1.695	2.424	2.888	3.794	4.845
29	1.681	2.404	2.864	3.763	4.805
30	1.668	2.385	2.841	3.733	4.768
35	1.613	2.306	2.748	3.611	4.611
40	1.571	2.247	2.677	3.518	4.493
45	1.539	2.200	2.621	3.444	4.399
50	1.512	2.162	2.576	3.385	4.323
75	1.428	2.042	2.433	3.197	4.084
100	1.383	1.977	2.355	3.096	3.954
200	1.304	1.865	2.222	2.921	3.731
500	1.243	1.777	2.117	2.783	3.555
1000	1.214	1.736	2.068	2.718	3.472
∞	1.150	1.645	1.960	2.576	3.291

Factors for One-Sided Tolerance Limits for Normal Distributions

Factors K_1 such that the probability is γ (gamma) that at least a proportion P of the distribution will be less than $\bar{Y}+K_1s$ or greater than $\bar{Y}-K_1s$.

	γ = 0.75				
n	**P=0.75**	**P=0.90**	**P=0.95**	**P=0.99**	**P=0.999**
3	1.464	2.501	3.152	4.396	5.805
4	1.255	2.134	2.681	3.726	4.911
5	1.152	1.962	2.463	3.421	4.507
6	1.088	1.859	2.336	3.244	4.273
7	1.043	1.790	2.250	3.126	4.119
8	1.010	1.740	2.188	3.042	4.008
9	0.985	1.701	2.141	2.977	3.924
10	0.964	1.671	2.104	2.927	3.858
11	0.947	1.645	2.073	2.885	3.804
12	0.932	1.624	2.048	2.851	3.760
13	0.920	1.606	2.026	2.822	3.722
14	0.909	1.591	2.007	2.797	3.690
15	0.899	1.577	1.991	2.775	3.661
16	0.891	1.565	1.976	2.756	3.636
17	0.883	1.554	1.963	2.739	3.614
18	0.876	1.545	1.952	2.723	3.595
19	0.870	1.536	1.941	2.710	3.577
20	0.864	1.528	1.932	2.697	3.560
21	0.859	1.521	1.923	2.685	3.546
22	0.854	1.514	1.915	2.675	3.532
23	0.849	1.508	1.908	2.665	3.520
24	0.845	1.502	1.901	2.656	3.508
25	0.841	1.497	1.895	2.648	3.497
26	0.838	1.492	1.889	2.640	3.487
27	0.834	1.487	1.883	2.633	3.478
28	0.831	1.483	1.878	2.626	3.469
29	0.828	1.478	1.873	2.620	3.461
30	0.825	1.475	1.869	2.614	3.454
35	0.813	1.458	1.849	2.588	3.421
40	0.803	1.445	1.834	2.568	3.395
45	0.795	1.434	1.821	2.552	3.375
50	0.788	1.425	1.811	2.538	3.358
∞	0.674	1.282	1.645	2.326	3.090

Factors for One-Sided Tolerance Limits for Normal Distributions

Factors K_1 such that the probability is γ (gamma) that at least a proportion P of the distribution will be less than $\bar{Y}+K_1s$ or greater than $\bar{Y}-K_1s$

	γ = 0.90				
n	**P=0.75**	**P=0.90**	**P=0.95**	**P=0.99**	**P=0.999**
3	2.603	4.258	5.311	7.340	9.651
4	1.972	3.188	3.957	5.438	7.129
5	1.698	2.742	3.400	4.666	6.111
6	1.540	2.494	3.092	4.243	5.556
7	1.435	2.333	2.894	3.972	5.202
8	1.360	2.219	2.754	3.783	4.955
9	1.302	2.133	2.650	3.641	4.771
10	1.257	2.066	2.568	3.532	4.629
11	1.219	2.011	2.503	3.443	4.514
12	1.188	1.966	2.448	3.371	4.420
13	1.162	1.928	2.402	3.309	4.341
14	1.139	1.895	2.363	3.257	4.273
15	1.119	1.867	2.329	3.212	4.215
16	1.101	1.842	2.299	3.172	4.164
17	1.085	1.819	2.272	3.137	4.119
18	1.071	1.800	2.249	3.105	4.078
19	1.058	1.782	2.227	3.077	4.042
20	1.046	1.765	2.208	3.052	4.009
21	1.035	1.750	2.190	3.028	3.979
22	1.025	1.737	2.174	3.007	3.952
23	1.016	1.724	2.159	2.987	3.927
24	1.007	1.712	2.145	2.969	3.903
25	1.000	1.702	2.132	2.952	3.882
26	0.992	1.691	2.120	2.937	3.862
27	0.985	1.682	2.109	2.922	3.843
28	0.979	1.673	2.099	2.909	3.826
29	0.973	1.665	2.089	2.896	3.810
30	0.967	1.657	2.080	2.884	3.794
35	0.942	1.624	2.041	2.833	3.730
40	0.923	1.598	2.010	2.793	3.679
45	0.907	1.577	1.986	2.761	3.638
50	0.894	1.559	1.965	2.735	3.604
∞	0.674	1.282	1.645	2.326	3.090

Factors for One-Sided Tolerance Limits for Normal Distributions

Factors K_1 such that the probability is γ (gamma) that at least a proportion P of the distribution will be less than $\bar{Y}+K_1s$ or greater than $\bar{Y}-K_1s$

	γ = 0.95				
n	P=0.75	**P=0.90**	**P=0.95**	**P=0.99**	**P=0.999**
3	3.806	6.155	7.656	10.553	13.857
4	2.618	4.162	5.144	7.042	9.214
5	2.150	3.407	4.203	5.741	7.502
6	1.895	3.006	3.708	5.062	6.612
7	1.732	2.755	3.399	4.642	6.063
8	1.618	2.582	3.187	4.354	5.688
9	1.532	2.454	3.031	4.143	5.413
10	1.465	2.355	2.911	3.981	5.203
11	1.411	2.275	2.815	3.852	5.036
12	1.366	2.210	2.736	3.747	4.900
13	1.328	2.155	2.671	3.659	4.787
14	1.296	2.109	2.614	3.585	4.690
15	1.268	2.068	2.566	3.520	4.607
16	1.243	2.033	2.524	3.464	4.535
17	1.220	2.002	2.486	3.414	4.471
18	1.201	1.974	2.453	3.370	4.415
19	1.183	1.949	2.423	3.331	4.364
20	1.166	1.926	2.396	3.295	4.318
21	1.152	1.905	2.371	3.263	4.277
22	1.138	1.886	2.349	3.233	4.239
23	1.125	1.869	2.328	3.206	4.204
24	1.114	1.853	2.309	3.181	4.172
25	1.103	1.838	2.292	3.158	4.142
26	1.093	1.824	2.275	3.136	4.115
27	1.083	1.811	2.260	3.117	4.089
28	1.075	1.799	2.246	3.098	4.066
29	1.066	1.788	2.232	3.080	4.043
30	1.058	1.777	2.220	3.064	4.022
35	1.025	1.732	2.167	2.995	3.934
40	0.999	1.697	2.126	2.941	3.866
45	0.978	1.669	2.092	2.898	3.811
50	0.960	1.646	2.065	2.862	3.766
∞	0.674	1.282	1.645	2.326	3.090

Factors for One-Sided Tolerance Limits for Normal Distributions

Factors K_1 such that the probability is γ (gamma) that at least a proportion P of the distribution will be less than $\bar{Y}+K_1 s$ or greater than $\bar{Y}-K_1 s$

	γ = 0.99				
n	P=0.75	**P=0.90**	**P=0.95**	**P=0.99**	**P=0.999**
6	2.848	4.411	5.406	7.335	9.550
7	2.491	3.859	4.728	6.412	8.346
8	2.253	3.497	4.285	5.812	7.564
9	2.083	3.241	3.972	5.389	7.015
10	1.954	3.048	3.738	5.074	6.606
11	1.853	2.898	3.556	4.829	6.288
12	1.771	2.777	3.410	4.633	6.035
13	1.703	2.677	3.290	4.472	5.827
14	1.645	2.593	3.189	4.337	5.652
15	1.595	2.522	3.102	4.222	5.504
16	1.552	2.459	3.028	4.123	5.377
17	1.514	2.405	2.963	4.037	5.265
18	1.481	2.357	2.905	3.960	5.167
19	1.450	2.314	2.854	3.892	5.080
20	1.423	2.276	2.808	3.832	5.001
21	1.399	2.241	2.766	3.777	4.931
22	1.376	2.209	2.729	3.727	4.867
23	1.355	2.180	2.694	3.681	4.808
24	1.336	2.154	2.662	3.640	4.755
25	1.319	2.129	2.633	3.601	4.706
26	1.303	2.106	2.606	3.566	4.660
27	1.287	2.085	2.581	3.533	4.618
28	1.273	2.066	2.558	3.502	4.579
29	1.260	2.047	2.536	3.473	4.542
30	1.247	2.030	2.516	3.447	4.508
35	1.195	1.957	2.430	3.334	4.564
40	1.154	1.902	2.364	3.249	4.255
45	1.121	1.857	2.312	3.180	4.168
50	1.094	1.821	2.269	3.125	4.096
∞	0.674	1.282	1.645	2.326	3.090

Distribution-Free Two-Sided Tolerance Limits, γ = 0.75 & 0.90

Values (r, s) such that we may assert with at least γ (gamma) confidence that 100P percent of the population lies between the r^{th} smallest and the s^{th} largest of a random sample of n.

	γ = 0.75				γ = 0.90			
n \ P	**0.75**	**0.90**	**0.95**	**0.99**	**0.75**	**0.90**	**0.95**	**0.99**
50	5, 5	2,1			5,4	1,1		
55	6, 6	2,2	1,1		5,5	2,1		
60	7,6	2,2	1,1		6,5	2,1		
65	7,7	3,2	1,1		6,6	2,2		
70	8,7	3,2	1,1		7,6	2,2		
75	8,8	3,3	1,1		7,7	2,2		
80	8,8	3,3	2,1		8,7	3,2	1,1	
85	10,9	4,3	2,1		8,8	3,2	1,1	
90	10,10	4,3	2,1		9,8	3,2	1,1	
95	11,10	4,3	2,1		9,9	3,3	1,1	
100	11,11	4,4	2,1		10,10	3,3	1,1	
110	12,12	5,4	2,2		11,11	4,3	2,1	
120	14,13	5,5	2,2		12,12	4,4	2,1	
130	15,14	6,5	3,2		13,13	5,4	2,1	
140	16,15	6,6	3,2		14,14	5,5	2,2	
150	17,17	6,6	3,3		16,15	5,5	2,2	
170	20,19	7,7	4,3		18,17	6,6	3,2	
200	23,23	9,8	4,4		21,21	8,7	3,3	
300	35,35	13,13	6,6	1,1	33,32	12,11	5,5	
400	47,47	18,18	9,8	2,1	45,44	16,16	8,7	1,1
500	59,59	23,22	11,11	2,1	57,56	21,20	10,9	1,1
600	72,71	28,27	13,13	2,2	68,68	26,25	12,11	2,1
700	84,83	33,32	16,15	3,2	80,80	30,30	14,14	2,2
800	96,96	37,37	18,18	3,3	92,92	35,34	16,16	3,2
900	108,108	42,42	21,20	4,3	104,104	40,39	19,18	3,2
1000	121, 120	47,47	23,22	4,4	117,116	44,44	21,20	3,3

When the values of r and s are not equal they are interchangeable.

Distribution-Free Two-Sided Tolerance Limits, γ = 0.95 & 0.99

Values (r, s) such that we may assert with at least γ (gamma) confidence that 100P percent of the population lies between the r^{th} smallest and the s^{th} largest of a random sample of n.

	γ = 0.95				γ = 0.99			
n \ P	**0.75**	**0.90**	**0.95**	**0.99**	**0.75**	**0.90**	**0.95**	**0.99**
50	4, 4	1,1			3, 3			
55	5, 4	1,1			4,3			
60	5,5	1,1			4,4			
65	6,5	2,1			5,4	1,1		
70	6,6	2,1			5,5	1,1		
75	7,6	2,1			5,5	1,1		
80	7,7	2,2			6,5	1,1		
85	8,7	2,2			6,6	2,1		
90	8,8	3,2			7,6	2,1		
95	9,8	3,2	1,1		7,7	2,1		
100	9,9	3,2	1,1		8,7	2,2		
110	10,10	3,3	1,1		9,8	2,2		
120	11,11	4,3	1,1		10,9	3,2		
130	13,12	4,4	2,1		11,10	3,3	1,1	
140	14,13	4,4	2,1		12,11	3,3	1,1	
150	15,14	5,4	2,1		13,13	4,3	1,1	
170	17,16	6,5	2,2		15,15	5,4	2,1	
200	20,20	7,6	3,2		18,18	6,5	2,2	
300	32,31	11,11	5,4		29,29	10,9	4,3	
400	43,43	15,15	7,6		40,40	14,13	6,5	
500	55,54	20,19	9,8	1,1	52,51	18,17	7,7	
600	67,66	24,24	11,10	1,1	63,63	22,22	9,9	
700	78,78	29,28	13,13	2,1	75,74	26,26	11,11	1,1
800	90,90	33,33	15,15	2,2	86,86	31,30	13,13	1,1
900	102,102	38,37	18,17	2,2	98,97	35,35	15,15	2,1
1000	114, 114	43,42	20,19	3,2	110, 109	40,39	18,17	2,1

When the values of r and s are not equal they are interchangeable.

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Distribution-Free One-Sided Tolerance Limits

Values (m) such that with at least γ (gamma) confidence that 100P percent of the population lies below the m^{th} largest (or above the m^{th} smallest) of a random sample of n.

	γ = 0.75				**γ = 0.90**				**γ = 0.95**				**γ = 0.99**			
n\P	**0.75**	**0.90**	**0.95**	**0.99**	**0.75**	**0.90**	**0.95**	**0.99**	**0.75**	**0.90**	**0.95**	**0.99**	**0.75**	**0.90**	**0.95**	**0.99**
50	10	3	1		9	2	1		8	2			6	1		
55	12	4	2		10	3	1		9	2			7	1		
60	13	4	2		11	3	1		10	2	1		8	1		
65	14	5	2		12	4	1		11	3	1		9	2		
70	15	5	2		13	4	1		12	3	1		10	2		
75	16	6	2		14	4	1		13	3	1		10	2		
80	17	6	3		15	5	2		14	4	1		11	2		
85	19	7	2		16	5	2		15	4	1		12	3		
90	20	7	2		17	5	2		16	5	1		13	3	1	
95	21	7	2		18	6	2		17	5	2		14	3	1	
100	22	8	2		20	6	2		18	5	2		15	4	1	
110	24	9	4		22	7	3		20	6	2		17	4	1	
120	27	10	4		24	8	3		22	7	2		19	5	1	
130	29	11	5		26	9	3		25	8	3		21	6	2	
140	31	12	5	1	28	10	4		27	8	3		23	6	2	
150	34	12	6	1	31	10	4		29	9	3		26	7	2	
170	39	14	7	1	35	12	5		33	11	4		30	9	3	
200	46	17	8	1	42	15	6		40	13	5		36	11	4	
300	70	26	12	2	65	23	10	1	63	22	9	1	58	19	7	
400	94	36	17	3	89	32	15	2	86	30	13	1	80	27	11	
500	118	45	22	3	113	41	19	2	109	39	17	2	103	35	14	1
600	143	55	26	4	136	51	23	3	133	48	21	2	126	44	18	1
700	167	65	31	5	160	60	28	4	156	57	26	3	149	52	22	2
800	192	74	36	6	184	69	32	5	180	66	30	4	172	61	26	2
900	216	84	41	7	208	79	37	5	204	75	35	4	195	70	30	3
1000	241	94	45	8	233	88	41	6	228	85	39	5	219	79	35	3